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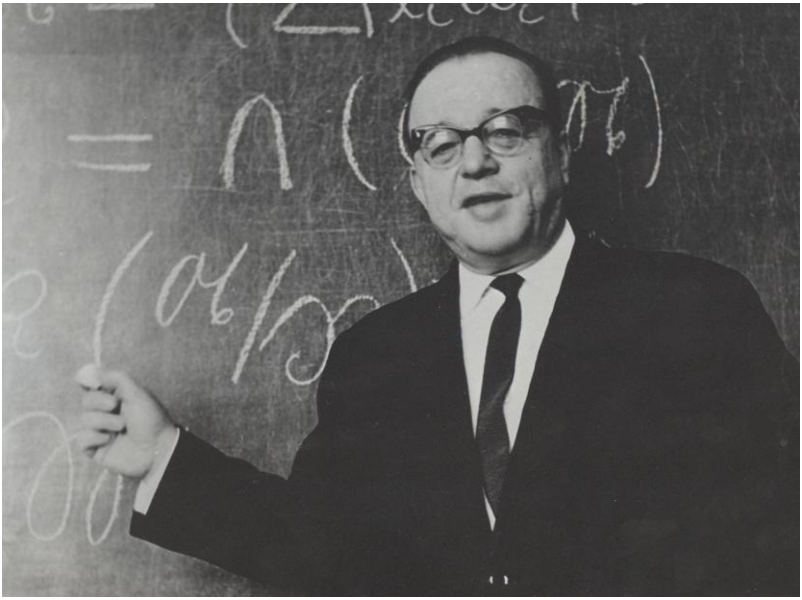
Vladimir Andrunakievich

(1917–1997)



Vladimir Andrunakievich, a prominent mathematician and great organizer of science, is known due to his remarkable results in the theory of radicals in rings and algebras, due to an algebraic school which he created in Moldova, and also due to his activities as one of the founders of the Academy of Sciences of Moldova and the Institute of Mathematics.

V. Andrunakievich was born on April 3, 1917. He finished the “Aleco Russo” lyceum in Kishinev in 1936 and graduated from the University of Iaşi in 1940. In 1940-1941 he worked as teacher of mathematics at a secondary school in Kishinev. During the World War II his family was evacuated to Kazakhstan (the town of Geambul). In 1943-

1947 he took postgraduate course at the Moscow State University, the great scientists O.Yu. Schmidt and A. G. Kurosh were his scientific supervisors.

In 1947 V. Andrunakievich successfully finished his postgradu-ate study and defended Doctor Thesis. He returned to Kishinev and worked as associated professor and then as full professor at the Kishinev State University and the Kishinev Pedagogical Institute (1947-1953). In 1953-1961 he worked at the Moscow Institute of Chemical Technologies. During this period he obtained a series of important results on radicals in rings, which constituted the bases of his Doctor habilitat thesis, defended in 1958.

When the Academy of Sciences was organized in Moldova, V.Andrunakievich returned to Kishinev, where he lived till the end of his life.

In 1961 V. Andrunakievich became a full member of the Academy of Sciences of Moldova and the director of the Institute of Physics and Mathematics. The Institute of Mathematics was organized in 1964, and V. Andrunakievich was its director for about 30 years. During this period he was a vice-president of the Academy of Sciences of Moldova (1965-1974, 1979-1990), the coordinator-academician of the Section of Physics and Mathematics of the ASM (1964-1985).

Scientific interests of Acad. V. Andrunakievich were concentrated on the theory of rings and algebras, he was among pioneers of a new domain, theory of radicals. His first cycle of works was devoted to the theory of radicals in associative rings and algebras. In 1946-1961 he distinguished from the class of all radicals the most useful in applications radicals, namely special and idempotent, and showed their usefulness by proving structural theorems. The ideas and constructions of this cycle helped to understand the connections between different special radicals which generalize the classical ones. Subidempotent radical named after V.Andrunakievich and the Andrunakievich's lemma have been used till present not only in the theory of associative rings, but also in other related algebraic systems.

A natural continuation of those investigations was presented in his second cycle of works devoted to the structural theory of rings and algebras. As long ago as in 1947 V. Andrunakievich showed the construction of so called adjoint fractions that was the first step to the structural theory of quasiregular algebras. In works of the period 1967-1972, V. Andrunakievich and his disciples developed the structural theory of rings and algebras without nilpotent elements, in its framework they found a logical completion of the classical Weierstrass, Dedekind and Krull theorems on decomposition of algebras without divisors of zero. These investigations were summarized in monograph [2].

The third cycle of Acad. V. Andrunakievich's works was connected with the generalization of noncommutative case of the classical theory of Notherian primarity, i.e. the additive theory of ideals. Principal goals of this theory are the proof of existence of the representation of an ideal as the intersection of some special ideals (primary, primal, tertiary, etc.), and also the uniqueness of such representation. In works of the period 1964-1972, V. Andrunakievich and his disciples showed that there is the unique generalization of primarity satisfying the existence and uniqueness conditions, and this generalization is the tertiarity. They developed the general additive theory which can be used not only for rings but also for many other algebraic systems.

V. Andrunakievich worked actively in many other domains having publications in topological algebra, on variaties of quasiregular and strongly regular algebras.

Mathematical talent of Acad. V. Andrunakievich was harmoniously combined with his distinguished teacher and organizer abilities. For his great contribution to the development of mathematics and education of high quality specialists Acad. V. Andrunakievich was awarded “Ordinul Republicii” and other high orders.

The organization of the Institute of Matematics of the ASM, the main centre of mathematical research in Moldova, and the creation of algebraic school in Moldova are his greatest merits.

The scientist of the world scale, V. Andrunakievich left a rich scientific inheritance of more than 150 published works. His ideas have been used actively and applied to different problems, his ar ticles have been permanently cited.

Respecting his memory, young algebraists continue the investigations being the content of V. Andrunakievich's life.

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Later developments related to some ideas of Andrunachievici: special radicals and The Lemma

Barry Gardner

(plenary invited speaker)

Abstract

The concept of *special radicals*, exemplified by the prime, nil, Jacobson and Brown-McCoy radicals, was central to Andrunachievici's seminal papers [1]. Subsequently questions posed in these papers have been answered, new ones have arisen, many answered in turn, and interesting open problems remain. In these papers the result which has become known as Andrunachievici's Lemma also appeared. In the intervening period versions of this lemma have been proved for various kinds of non-associative rings. We shall give an account of these developments.

Keywords: algebra, radical theory.

1 Introduction

We shall use the terms *radical class* and, more briefly, *radical*, in the sense of Kurosh and Amitsur: a radical class of rings is a non-empty homomorphically closed class $\mathcal{R}$ such that for every ring A,

$$\mathcal{R}(A) = \sum\{I \triangleleft A : I \in \mathcal{R}\} \in \mathcal{R}$$

and $\mathcal{R}(A/\mathcal{R}(A)) = 0$. Associated with a radical $\mathcal{R}$ is the class $\mathcal{S}$ of $\mathcal{R}$-*semi-simple* rings: the rings A for which $\mathcal{R}(A) = 0$. In general a class is called a *semi-simple class* if it is associated in this way with some radical. Semi-simple classes are intrinsically characterized as non-empty classes which are hereditary (closed under ideals) closed under extensions and closed under subdirect products. Each class $\mathcal{C}$ is contained in a smallest radical class $L(\mathcal{C})$ called the *lower radical class* defined by $\mathcal{C}$. There is also a largest radical class $U(\mathcal{C})$ for which all rings in $\mathcal{C}$ are semi-simple, called the *upper radical class* defined by $\mathcal{C}$.

2 Special radicals

A class of prime rings is *special* if it is hereditary for non-zero ideals and closed under essential extensions. (A ring A is an *essential extension* of an ideal I if I has non-zero intersection with every non-zero ideal of A.) In [1], a special radical was defined to be a radical which is the upper radical defined by some special class of prime rings. For example the prime radical is the upper radical defined by the special class of all prime rings and the Jacobson radical is defined by the special class of (left) primitive rings. It was shown that *a radical class is special if and only if it contains all nilpotent rings, is hereditary and all its semi-simple rings are subdirect products of semi-simple prime rings.* It has since been shown by Beidar [2] that the hereditary assumption is unnecessary.

One of the questions asked in [1] was whether a hereditary radical containing all nilpotent rings must be special. The first counterexample was constructed by Ryabukhin [3]: *the upper radical defined by the class of boolean rings without ideals isomorphic to the field Z_2 is not special.* Many examples are now known ([4], [5], [6], [7], [8] among others) and many of these have a family resemblance to this first one: they are upper radical classes defined by rings whose prime homomorphic images can't be isomorphic to ideals. France-Jackson [8] proved that *a radical, other than the prime radical, for which all prime essential rings are semi-simple is not special.*

Snider [9] observed that radical classes form a complete lattice (a "large" lattice, as they don't form a set), as do the special radicals.

In the special case the operations are: $\bigwedge \mathcal{R}_i = \bigcap \mathcal{R}_i$ and $\bigvee \mathcal{R}_i =$ the smallest special radical containing all the $\mathcal{R}_i$. (There *is* a smallest special radical class containing a given class $\mathcal{C}$ of rings: we'll denote it by $L_S(\mathcal{C})$, or $L_S(A)$ if $\mathcal{C} = \{A\}$.) Much work has been done on characterizing the *atoms* of this lattice. In [4] it was shown that if T is an idempotent simple ring then $L_S(T)$ is an atom. The next examples were due to Korolczuk [10], using what she called $*$-rings: prime rings whose proper homomorphic images are in the prime radical class. She showed that *if A is a $*$-ring, then $L_S(A)$ is an atom.* This is a generalization, as idempotent simple rings are clearly $*$-rings.

Going from small to large, the *coatoms* of special radicals were characterized by Krachilov [14]: *$\mathcal{R}$ is a special coatom if and only if $\mathcal{R} = U(T)$ where T is a full matrix ring over a finite field.*

Every prime ring A is contained in a smallest special class π_A[15]. This consists of all prime rings having an ideal isomorphic to an accessible subring of A. The smaller the special class, the larger the corresponding special radical, so we can stay up towards the large end of the scale by looking at small special classes, and a natural question is: for which prime rings A is π_A the set of ideals of A. There is an error in [16] which deals with this question, and the complete answer is not known. The best we have at the moment is the following.

Let A be a prime ring with the property that every isomorphism between ideals is the restriction of an automorphism. Then π_A is the set of ideals of A if and only if A satisfies the following.

(i) A is a simple ring with identity of characteristic 0, or

(ii) $pA = 0$ for some prime and A is simple with identity or a certain algebra over Z_p which is not an algebra over any other field, or

(iii) A is additively torsion-free and reduced and a principal ideal domain such that each proper homomorphic image is isomorphic to some Z_n.

In [1] a special radical was called a *dual special radical* if it is the upper radical defined by a (special) class of subdirectly irreducible rings. In [17] an intermediate type of radical was introduced. If $\mathcal{R}$ is a special radical with semi-simple class $\mathcal{S}$, a ring $B \in \mathcal{S}$ is said to be $\mathcal{S}$-*subdirectly* *irreducible* if $\bigcap\{J \triangleleft B : B/J \in \mathcal{S}\} \neq 0$. We say that R is *extraspecial* if every ring in $\mathcal{S}$ is a subdirect product of $\mathcal{S}$-subdirectly irreducible rings. Clearly dual special $\Rightarrow$ extraspecial $\Rightarrow$ special. Neither implication is reversible. The first example of a special but not extraspecial radical was given by Beidar [18]. It is not known whether any of the "standard" radicals is extraspecial (apart from the dual ones). France-Jackson [19] has shown that the prime radical is extraspecial if and only if the lattice of special radicals is atomic.

3 The Lemma

The result from [1] which is now known as the Andrunachievici lemma is the following.

If $I \triangleleft J \triangleleft A$ and I^* is the ideal of A generated by I, then $(I^*/I)^3 = 0$. This is a very useful result in radical theory and ring theory, and some generalizations have been found. For *alternative rings* Hentzel and Slater [20] showed that $\bigcap(I^*/I)^n = 0$. Then (modulo some difficulty with 2- and 3-torsion) Pchelintsev [21] showed that $(I^*/I)^n = 0$ with $n \leq 4 \cdot 5^6$. Hentzel [22] reduced the bound to 4. Medvedev [23] proved a version for Jordan rings.

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Section 1

Algebra and Logic

Matrix Wreath Products

Efim Zelmanov

(plenary invited speaker)

Abstract

The talk will focus on a new construction of matrix wreath product of algebras that is analogous to wreath product of groups. To prove its usefulness we will discuss applications to embedding theorems and growth functions of algebras.

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Groebner-Shirshov bases methods for GDN algebras and PBW type theorems

L.A.Bokut*, Yuqun Chen, Zerui Zhang

(*plenary invited speaker)

Abstract

We review some of our recent results for Gelfand-Dorfman-Novikov (GDN) algebras invented independently by I.M.Gelfand – I.Ya. Dorfman (Funk. Anal. App., 1979) and A.A. Balinskii – S.P.Novikov (Doklady AN USSR,1985). We are dealing with GDN (super)algebras and GDN-Poisson algebras.

Keywords: Groebner-Shirshov bases, GDN (super)algebras, GDN-Poisson algebras, PBW type theorems.

1 Definitions

Definiton 1. A left *GDN algebra* $\mathcal{A}$ is a vector space with a binary linear operation $\circ$ satisfying the two identities

$$x \circ (y \circ z) - (x \circ y) \circ z = y \circ (x \circ z) - (y \circ x) \circ z \quad \text{(left symmetry)},$$

$$(x \circ y) \circ z = (x \circ z) \circ y \quad \text{(right commutativity)}.$$

Definiton 2. A *GDN superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ with a bilinear operation $\circ$ satisfying

$$x \circ (y \circ z) - (x \circ y) \circ z = (-1)^{|x||y|}(y \circ (x \circ z) - (y \circ x) \circ z) \text{ (left symmetry)},$$

$$(x \circ y) \circ z = (-1)^{|y||z|}(x \circ z) \circ y \text{ (right commutativity)}$$

for any $x, y, z \in \mathcal{A}_0 \cup \mathcal{A}_1$.

Definiton 3. A *GDN-Poisson algebra* $\mathcal{A}$ is a vector space with two bilinear operations "$\cdot$" and "$\circ$" such that $(\mathcal{A}, \cdot)$ forms a commutative associative algebra and $(\mathcal{A}, \circ)$ forms a GDN algebra with the compatibility conditions:

$$(x \cdot y) \circ z = x \cdot (y \circ z),$$

$$(x \circ y) \cdot z - x \circ (y \cdot z) = (y \circ x) \cdot z - y \circ (x \cdot z), \; x, y, z \in \mathcal{A}.$$

We only consider GDN-Poisson algebras with unit e with respect to $\cdot$.

Definiton 4. A *commutative associative differential superalgebra* is a $\mathbb{Z}_2$-graded associative algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with a linear derivation D of degree 0, which satisfies

$$D(u \cdot v) = Du \cdot v + u \cdot Dv, \; u \cdot v = (-1)^{|u||v|} v \cdot u,$$

for any $u, v \in \mathcal{A}_0 \cup \mathcal{A}_1$.

Definiton 5. A special GDN-Poisson admissible $(\mathcal{A}, \cdot, *, D)$ *is a vector space with bilinear operations* $\cdot, *$ *and a linear operation* D *such that* $(\mathcal{A}, \cdot)$ *forms a commutative associative algebra with unit* e, $(\mathcal{A}, *)$ *forms a commutative associative algebra and* "$\cdot, *, D$" *are compatible in the sense that the following identities hold*:

$$\begin{aligned}
&(x \cdot y) * z = x \cdot (y * z), \\
&D(x * y) = (Dx) * y + x * (Dy), \\
&D(x \cdot y) = (Dx) \cdot y + x \cdot (Dy) - x \cdot y \cdot (De).
\end{aligned}$$

2 Linear bases and PBW type theorems

Definition 6. (see [4] for right GDN algebras) We call w a *GDN tableau* over a well-ordered set $X = X_1 \cup X_0$, if

$$w = (\dots((a_{1,r_1+1} \circ A_1) \circ A_2) \circ \cdots \circ A_n) \text{ (left-normed bracketing)},$$

where $A_i = (a_{i,r_i} \circ \cdots \circ (a_{i,3} \circ (a_{i,2} \circ a_{i,1})) \dots)$ (right-normed bracketing), $1 \leq i \leq n, \; a_{i,j} \in X$, satisfying the following relations:

(i) $r_i \geq r_{i+1}$,
(ii) $a_{i,1} \geq a_{i+1,1}$ if $r_i = r_{i+1}$,
(iii) $a_{1,r_1+1} \geq \cdots \geq a_{1,2} \geq a_{2,r_2} \geq \cdots \geq a_{2,2} \geq \cdots \geq a_{n,r_n} \geq \cdots \geq a_{n,2}$.

Definition 7. We call w a *GDN supertableau* if w is a GDN tableau which also satisfies
(iv) $a_{i,j} \neq a_{t,l}$, if $a_{i,j}, a_{t,l} \in X_1$ and $j, l \geq 2$,
(v) $a_{i,1} \neq a_{i+1,1}$, if $r_i = r_{i+1}$ and $a_{i,1}, a_{i+1,1} \in X_1$.

Definition 8. Let $[X]$ be the commutative monoid generated by X with unit e. We call $T = u \cdot w$ a *GDN-Poisson tableau* over X, if $u = b_1 \cdots b_m \in [X]$ (each $b_i \in X$) and $w = (\cdots((a_{1,r_1+1} \circ A_1) \circ A_2) \circ \cdots \circ A_n)$ is a GDN tableau over $X \cup \{e\}$ ($e < x$ for any $x \in X$) satisfying:
(i) $a_{n,2} \geq b_1 \geq \cdots \geq b_m$, (ii) if $a_{n,2} = e$, then $m = 0$, i.e., $T = w$.

Theorem 1 [1]. (PBW type theorem in Shirshov form) Let $GDN(X)$ be a free Gelfand-Dorfman-Novikov algebra, $k\{X\}$ be a free commutative differential algebra, $S \subseteq GDN(X)$ and S^c a Gröbner-Shirshov basis in $k\{X\}$. Then

(i) $S' = \{uD^m s \mid s \in S^c, u \in [D^\omega X], m \in \mathbb{N}, wt(u\overline{D^m s}) = -1\}$ is a Gröbner-Shirshov basis in $GDN(X)$.

(ii) The set $Irr(S') = \{w \in [D^\omega X] \mid w \neq u\overline{D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S^c, wt(w) = -1\} = GDN(X) \cap Irr[S^c]$ is a linear basis of $GDN(X|S)$. Thus, any Gelfand-Dorfman-Novikov algebra $GDN(X|S)$ is embeddable into its universal enveloping commutative differential algebra $k\{X|S\}$.

Theorem 2 [2]. The set of the GDN-Poisson tableaux over X forms a linear basis of the free GDN-Poisson algebras.

Theorem 3 [2]. Any GDN-Poisson algebras can be embedded into its universal enveloping special GDN-Poisson admissible algebras.

Theorem 4 [2]. Any GDN-Poisson algebras $\mathcal{A}$ satisfying the identity $x \circ (y \cdot z) = (x \circ y) \cdot z + (x \circ z) \cdot y$ is isomorphic to a commutative associative differential algebras.

Theorem 5 [3]. The set of all the GDN supertableaux over $X =$

$X_0 \cup X_1$ forms a linear basis of the free GDN superalgebra if the characteristic of the field is not 2.

Theorem 6 [3]. Any GDN superalgebra can be embedded into its universal enveloping commutative associative differential superalgebra.

Theorem 7 [3]. Let $\mathcal{A} = GDN(X_0 \cup X_1|R)_{sup} = \mathcal{A}_0 \oplus \mathcal{A}_1$, where $R \subset GDN(X_0 \cup X_1)_{sup}$, $|X_1| < \infty$. Suppose there exists a natural number n such that for any $a \in \mathcal{A}_0$, $((\cdots((a \circ a) \circ a) \cdots a) \circ a) = 0$ (n times). Then there exists $q \in \mathbb{Z}_{\geq 0}$ such that $((\cdots((\mathcal{A} \circ \mathcal{A}) \circ \mathcal{A}) \cdots \mathcal{A}) \circ \mathcal{A}) = 0$ (q times).

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Arithmetic of π_0-critical module

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Abstract

In this paper, for a specific kind of one-dimensional formal groups over the ring of integers of a local field in the case of small ramification we study the arithmetic of the formal module constructed on the maximal ideal of a local field, containing all the roots of the isogeny. This kind of formal groups is a little broader than Honda groups. The Shafarevich system of generators is constructed.

Keywords: formal modules, local fields.

Let K_0 be a local field (a finite extension of $\mathbb{Q}_p$) with the ring of integers $\mathcal{O}_0$, and a prime element π_0; let $q = p^f$ be the cardinality of the residue field $\overline{K}_0$; let K be a finite extension of the field K_0 with the ring of integers $\mathcal{O}$ and a prime element π, e_0 the ramification index of K/K_0; let N be the inertia subfield of K/K_0 with the ring of integers $\mathcal{O}_N$, and let σ denote the Frobenius automorphism of N/K_0.

We denote by $\mathcal{O}_N[[\Delta]]'$ the non-commutative ring of 'power series' $\sum a_i\Delta^i$, $a_i \in \mathcal{O}_N$ with multiplication rule

$$\Delta a = \sigma(a)\Delta.$$

There is a natural operation $(u, f) \mapsto u \circ f$ by the elements u of $\mathcal{O}_N[[\Delta]]'$ on the elements f of $N[[X]]$, defined by equality:

$$(a\Delta)(\sum c_i X^i) = \sum a c_i^\sigma X^{qi},\ a \in \mathcal{O}_N[[\Delta]]'$$

and this turns $N[[X]]$ into $\mathcal{O}_N[[\Delta]]'$-module. $\mathcal{O}[[\Delta]]$ also has natural structure of a left $\mathcal{O}_N[[\Delta]]'$-module.

From now on we assume that $e_0 < q$.

It can be shown that each formal $\mathcal{O}_0$-module over the ring $\mathcal{O}$ is strictly isomorphic to an $\mathcal{O}_0$-typical one, the logarithm of which can be represented in the form $\lambda(X) = \Lambda(\Delta)(X)$, where

$$\Lambda = vu^{-1}, \tag{1}$$
$$v \in \mathcal{O}[\Delta], \deg v < h+1,$$
$$u = \pi_0 - a_h B\Delta^h, B \in 1 + \mathcal{O}_N[[\Delta]]\Delta.$$

The pair (u, v) is said to be the type of formal $\mathcal{O}_0$-module.

Definition 1. *Formal $\mathcal{O}_0$-module with height h is π_0-critical if its type (u, v) satisfies the condition*

$$v \equiv 0 \mod (\pi_0, \Delta^h)$$

We will consider only this kind of formal $\mathcal{O}_0$-modules. For them, and only for them, homomorphism $[\pi_0]$ can be expressed as

$$[\pi_0](X) = \pi_0 X + \pi_0 f(X)X^2 + c_h X^{q^h} + \ldots, c_h \in \mathcal{O}_K^*,$$

where $f(X)$ is a polynomial of degree $q^h - 3$ with integral coefficients.

Let F be a π_0-critical formal $\mathcal{O}_0$-module of type (u, v). Then $v(\Delta) = \pi_0 - \pi r_1 - \ldots - \pi^{e_0 - 1} r_{e_0 - 1}$, where $r_i = \sum_{j=1}^{h} \rho_j^{(i)} \Delta^j$, $0 < i < e_0$.

Let $\lambda = \sum_{i=0}^{e_0-1} \pi^i \lambda_i(X)$, $\lambda_i(X) \in \mathcal{O}_N[[X]]$ and $\mu = \sum_{i=0}^{e_0-1} \pi^i \mu_i(X)$, where

$$\mu_0(X) = B\lambda_0^{\sigma^h}(X);$$
$$\mu_i(X) = -a_h^{-1}\rho_h^{(i)} \left(\lambda_0^{\sigma^h}(X) - \frac{a_h^{\sigma^h} \mu_0^{\sigma^h}}{\pi_0}(X^{q^h}) \right) - \frac{a_h^{-1} r_i a_h \mu_0(X)}{\pi_0},$$
$$0 < i < e_0. \tag{2}$$

The series $\mu_i(X)$ depends only on μ_0 and λ_0.

Theorem 1. *Let F be a formal $\mathcal{O}_0$-module over $\mathcal{O}$ of type (u, v) with logarithm $\lambda(X)$. Then*

1. *μ is the logarithm of a π_0-critical formal $\mathcal{O}_0$-module F_1 of type (u_1, v_1), where $u_1 = u^{\sigma^h} B^{-1}$, $v_1 = a_h^{-1} v a_h$.*
2. *$f := [\frac{\pi_0}{a_h}]_{F,F_1} = \mu^{-1}(\frac{\pi_0}{a_h}\lambda)$ is a homomorphism from F to F_1 and $f \equiv X^{q^h} \mod \pi_0$.*

So, for π_0-critical formal groups we also have a surjective (but not injective) operator $\mathcal{A} : F \mapsto F_1$ and a sequence of homomorphisms f_m:

$$F \xrightarrow{f} F_1 \xrightarrow{f_1} F_2 \to \cdots \to F_{n-1} \xrightarrow{f_{n-1}} F_N.$$

We denote by L a finite extension of K which contains $\mathrm{Ker}[\pi_0^n]$, with ring of integers $\mathcal{O}_L$, maximal ideal $\mathfrak{M}$, prime element Π. Let T be the inertia subfield in L/K_0, with ring of integers $\mathcal{O}_T$, and $\mathfrak{R}$ the Teichmüller set of representatives in $\mathcal{O}_T$; and e the index of ramification of L/K_0. The group law F induces on $\mathfrak{M}$ the structure of formal $\mathcal{O}_0$-module $F(\mathfrak{M})$.

Let $g : \mathcal{O}_N[[t]] \to \mathcal{O}_K$ be a ring homomorphism, mapping t to π. There exists a formal $\mathcal{O}_0[[t]]$-module $F_t(X,Y)$, such that $g_*(F_t(X,Y)) = F(X,Y)$ (g_* applies g to the coefficients). Suppose $U(\Delta)$ is a type of F_t and λ_t is its logarithm. Define the action Δ on the series from $\mathcal{O}_T[[X]][[t]]X$ as follows:

$$\Delta\left(\sum a_{ij} X^i t^j\right) = \sum a_{ij}^{\sigma} X^{qi} t^{qj}.$$

The Artin-Hasse maps E_{F_t} and l_{F_t} are given by:

$$E_{F_t}(f) = \lambda_{F_t}^{-1}(\pi_0 U(\Delta)^{-1} f),$$
$$l_{F_t}(f) = \pi_0^{-1} U(\Delta) \lambda_{F_t}(f).$$

An element $\omega \in F(\mathfrak{M})$ is called primary (π_0^n-primary) if the extension $L(\widetilde{\omega})/L$, where $[\pi_0^n]_F(\widetilde{\omega}) = \omega$, is unramified.

Let $a \in \mathcal{O}_T$, $\xi \in \mathrm{Ker}[\pi_0^n] \setminus \mathrm{Ker}[\pi_0^{n-1}]$, $\xi = \sum_{i=1}^{\infty} \alpha_i \Pi^i$, $\underline{\xi}(X) = \sum_{i=1}^{\infty} \alpha_i X^i \in \mathcal{O}_T[[X]]$. In [3] it was proved that the element

$$P_F(a) = E_{F_t}\left(\pi_0^{n-1}(\sum_{i>0} \alpha_i a_i \Delta^i)\lambda_{F_t}(\underline{\xi}(X))\right)|_{X=\Pi,\, t=\pi},$$

where $a_i = a + a^\sigma + \ldots + a^{\sigma^{i-1}}$ is π_0^n-primary in $F(\mathfrak{M}_L)$.

Let G_0 and $G_\rho, 0 < \rho < \mathbf{f}h$ be formal $\mathcal{O}_0$-modules, strictly isomorphic to $\mathcal{A}^{n-1}F$, for which $g_0(x) = \pi_{n-1}x + x^{q^h}$ and $g_\rho(x) = \pi_{n-1}x + \pi_{n-1}x^{p^\rho} + x^{\rho^h}$ are distinguished isogenies. Let $\mathcal{E}_n^0$, $\mathcal{E}_n^\rho$ be the isomorphisms $\mathcal{A}G_0 \longrightarrow \mathcal{A}^nF$ and $\mathcal{A}G_\rho \longrightarrow \mathcal{A}^nF$ respectively.

Theorem 2. *The elements*

$$\{P_F(a), \mathcal{E}_n^0(\theta\Pi^i), \mathcal{E}_n^\rho(\theta\Pi^i),\, \theta \in \mathfrak{R}, 0 < i < \frac{eq^h}{q^h - 1} + 1,\, q^h \nmid i\} \tag{3}$$

form a system of generators of the $\mathcal{O}_0$-module $F(\mathfrak{M}_L)$.

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Semidirect product of pseudonormed rings and semi-isometric isomorphism

S.A. Aleschenko

Abstract

We determine the pseudonorm for the semidirect product of pseudonormed rings. We study sufficient conditions for keeping a semi-isometric isomorphism (semi-isometric on the left, semi-isometric on the right) when taking a semidirect product of pseudonormed rings.

Keywords: pseudonormed rings, quotient rings, ideal, isometric homomorphism, semi-isometric isomorphism, semidirect product of pseudonormed rings.

We will consider that a pseudonormed ring is a ring R which may be non-associative and has a pseudonorm (see [1], definition 2.3.1).

The following theorem on isomorphism is often applied in algebra and, in particular, in the ring theory:

Theorem 1. If A is a subring of a ring R and I is an ideal of the ring R, then the quotient rings $A/(A \cap I)$ and $(A+I)/I$ are isomorphic rings.

In particular, if $A \cap I = 0$, then the ring A is isomorphic to the ring $(A+I)/I$, i.e. the rings A and $(A+I)/I$ possess identical algebraic properties.

Since it is necessary to take into account properties of pseudonorms when studying the pseudonormed rings, then one needs to consider isomorphisms which keep pseudonorms. Such isomorphisms are called isometric isomorphisms.

Theorem 1 does not always take place for pseudonormed rings. As it's shown in Theorem 2.1 from [2], it is impossible to tell anything more than the validity of the inequality $\bar{\xi}\left(\varphi(r)\right)\le\xi(r)$ in case $A\bigcap I=0$.

The case when A is an ideal of a pseudonormed ring (R,ξ) was studied in [2], the case when A is a one-sided ideal of a pseudonormed ring (R,ξ) was studied in [3].

The following notions were introduced in [2] and [3]:

Definition 1. Let (R,ξ) and $\left(\bar{R},\bar{\xi}\right)$ be pseudonormed rings. An isomorphism $\varphi: R\to\bar{R}$ is called a semi-isometric isomorphism (a semi-isometric on the left, a semi-isometric on the right) if there exists a pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$ such that the following conditions are valid:

1) the ring R is an ideal (a left ideal, a right ideal) in the ring $\hat{R}$;

2) $\hat{\xi}(r)=\xi(r)$ for any $r\in R$;

3) the isomorphism φ can be extended up to an isometric homomorphism $\hat{\varphi}:\left(\hat{R},\hat{\xi}\right)\to\left(\bar{R},\bar{\xi}\right)$ of the pseudonormed rings, i.e. $\bar{\xi}\left(\hat{\varphi}(\hat{r})\right)=\inf\left\{\xi(\hat{r}+a)\middle| a\in\ker\hat{\varphi}\right\}$ for all $\hat{r}\in\hat{R}$.

The following theorems were proved in [2] and [3]:

Theorem 2. Let (R,ξ) and $\left(\bar{R},\bar{\xi}\right)$ be pseudonormed rings and $\varphi: R\to\bar{R}$ be a ring isomorphism. The isomorphism $\varphi:(R,\xi)\to\left(\bar{R},\bar{\xi}\right)$ is a semi-isometric isomorphism iff the inequalities $\dfrac{\xi(a\cdot b)}{\xi(b)}\le\bar{\xi}\left(\varphi(a)\right)\le\xi(a)$ and $\dfrac{\xi(a\cdot b)}{\xi(a)}\le\bar{\xi}\left(\varphi(b)\right)\le\xi(b)$ are true for any $a,b\in R\setminus\{0\}$.

Theorem 3. Let (R,ξ) and $\left(\bar{R},\bar{\xi}\right)$ be pseudonormed rings and $\varphi: R\to\bar{R}$ be a ring isomorphism. The isomorphism $\varphi:(R,\xi)\to\left(\bar{R},\bar{\xi}\right)$ is a semi-isometric isomorphism on the left (on the right) iff the inequalities $\xi(a \cdot b) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)$ $\left(\xi(b \cdot a) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)\right)$ and $\bar{\xi}(\varphi(a)) \leq \xi(a)$ are true for any $a, b \in R$.

This paper is a continuation of [2] and [3] and it is devoted to the study of keeping a semi-isometric isomorphism on a semidirect product of pseudonormed rings.

Let R and S be rings and on S be defined a multiplication operation of elements of S by elements of R from the left and the right such that the group $S(+)$ becomes a right and a left R-module. Let Q be the direct sum of the groups $R(+)$ and $S(+)$. Let's define on Q a multiplication operation:

$$q_1 \cdot q_2 = \left(s_1 \cdot s_2 + r_1 \cdot s_2 + s_1 \cdot r_2, r_1 \cdot r_2\right)$$

for any $s_1, s_2 \in S$ and $r_1, r_2 \in R$. It's easy to see that Q is a ring.

Proposition 1. Let (R, ξ) and (S, η) be pseudonormed rings and the inequalities $\eta(s \cdot r) \leq \eta(s) \cdot \xi(r)$ and $\eta(r \cdot s) \leq \xi(r) \cdot \eta(s)$ are fulfilled for any $s \in S$ and $r \in R$. Then the function $\zeta(s, r) = \eta(s) + \xi(r)$ is a pseudonorm on the ring Q.

The ring (Q, ζ) is called a semidirect product of pseudonormed rings (S, η) and (R, ξ).

Theorem 4. Let (Q, ζ) be a semidirect product of pseudonormed rings (S, η) and (R, ξ), $(\bar{Q}, \bar{\zeta})$ be a semidirect product of pseudonormed rings $(\bar{S}, \bar{\eta})$ and $(\bar{R}, \bar{\xi})$. Let $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ and $\psi : (S, \eta) \to (\bar{S}, \bar{\eta})$ be semi-isometric isomorphisms on the left such that the inequalities $\eta(s \cdot r) \leq \bar{\eta}(\psi(s)) \cdot \xi(r)$ and $\eta(r \cdot s) \leq \bar{\xi}(\varphi(r)) \cdot \eta(s)$ are fulfilled for any $s \in S$ and $r \in R$. Then the mapping $\omega : (Q, \zeta) \to (\bar{Q}, \bar{\zeta})$ given by $\omega((s, r)) = (\psi(s), \varphi(r))$ is a semi-isometric isomorphism on the left.

Theorem 5. Let (Q, ζ) be a semidirect product of pseudonormed rings (S, η) and (R, ξ), $(\bar{Q}, \bar{\zeta})$ be a semidirect product of pseudonormed rings $(\bar{S},\bar{\eta})$ and $(\bar{R},\bar{\xi})$. Let $\varphi:(R,\xi)\to(\bar{R},\bar{\xi})$ and $\psi:(S,\eta)\to(\bar{S},\bar{\eta})$ be semi-isometric isomorphisms on the right such that the inequalities $\eta(s\cdot r)\le\eta(s)\cdot\bar{\xi}(\varphi(r))$ and $\eta(r\cdot s)\le\xi(r)\cdot\bar{\eta}(\psi(s))$ are fulfilled for any $s\in S$ and $r\in R$. Then the mapping $\omega:(Q,\zeta)\to(\bar{Q},\bar{\zeta})$ given by $\omega((s,r))=(\psi(s),\varphi(r))$ is a semi-isometric isomorphism on the right.

Theorem 6. Let (Q,ζ) be a semidirect product of pseudonormed rings (S,η) and (R,ξ), $(\bar{Q},\bar{\zeta})$ be a semidirect product of pseudonormed rings $(\bar{S},\bar{\eta})$ and $(\bar{R},\bar{\xi})$. Let $\varphi:(R,\xi)\to(\bar{R},\bar{\xi})$ and $\psi:(S,\eta)\to(\bar{S},\bar{\eta})$ be semi-isometric isomorphisms such that the inequalities $\eta(s\cdot r)\le\bar{\eta}(\psi(s))\cdot\bar{\xi}(\varphi(r))$ and $\eta(r\cdot s)\le\bar{\xi}(\varphi(r))\cdot\bar{\eta}(\psi(s))$ are fulfilled for any $s\in S$ and $r\in R$. Then the mapping $\omega:(Q,\zeta)\to(\bar{Q},\bar{\zeta})$ given by $\omega((s,r))=(\psi(s),\varphi(r))$ is a semi-isometric isomorphism too.

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Properties of the lattice of ring topologies

V.I. Arnautov, G.N. Ermakova

Abstract

A nilpotent ring $\widetilde{R}$ and two ring topologies $\widetilde{\tau}''$ and τ^* on $\widetilde{R}$ are constructed such that τ^* is a coatom in lattice of all ring topologies of the ring $\widetilde{R}$ and such that between $\inf\{\widetilde{\tau}'', \tau_d\}$ and $\inf\{\widetilde{\tau}'', \tau^*\}$ there exists an infinite chain of ring topologies, where τ_d is the discrete topology.

Keywords: nilpotent ring, ring topology, lattice of ring topologies, unrefinable chains, coatoms, infimum of ring topologies.

1 Introduction

After Markov A.A. has investigated in 1946 the question of the existence of nondiscrete Hausdorff group topologies on infinite groups, a similar question naturally appeared for other algebraic systems, and in particular for infinite rings.

The present paper is devoted to the study of the lattice of ring topologies, and consists of two parts. An overview of some known results is given in the first part of the paper. The new result of the paper is an example that between $\inf\{\widetilde{\tau}'', \tau_d\}$ and $\inf\{\widetilde{\tau}'', \tau^*\}$ there exists an infinite chain of ring topologies, where τ_d is the discrete topology (see Theorem 4 of the second part of the paper).

2 Review of previously obtained results

A method of specifying such topologies on countable groups was indicated, a similar question naturally appeared for other algebraic systems, and in particular for infinite rings.

Definition 1. As usual, a partially ordered set $(M,<)$ is called a lattice, if there are $\inf\{a,b\}$ and $\sup\{a,b\}$ for any elements $a,b\in M$.

Definition 2. As usual, a lattice $(M,<)$ is called modular if $\inf\{\sup\{a,b\},c\}=\sup\{a,\inf\{b,c\}\}$ for any elements $a,b,c\in M$ such that $a\le c$.

It is known that for any ring the lattice of all ring topologies has the form:

● discrete topology
.... coatoms
....
....
....
.... atoms
● antidiscrete topology

Moreover, for any nonzero ring (i.e. $R\neq\{0\}$) the lattice contains a discrete topology and an antidiscrete topology. In this case, the antidiscrete topology is a coatom.

It is clear that for any finite simple ring the lattice has only two topologies (discrete and antidiscrete topologies).

An example of an infinite ring for which the lattice of all ring topologies contains only two topologies was constructed in [3].

Theorem 1. [4].For any countable ring, the lattice of all ring topologies contains two in the degree of the continuum coatoms, in each of which the topological ring is a Hausdorff space.

Definition 3. As usual, a chain $a_1<a_2<\cdots<a_n$ of elements of a partially ordered set $(M,<)$ is called unrefinable if for any $1\le i\le n-1$ between elements a_i and a_{i+1} there are no other elements.

It is known the following result.

Theorem 2. If $(M,<)$ is a modular lattice and $a_1 < a_2 < \cdots < a_n$ is a finite unrefinable chain of elements, then $k \le n$ for any chain $a_1' < a_2' < \cdots < a_k'$ of elements of the set M, such that **:**

2.1. $a_1 = a_1'$ and $a_n = a_k'$;

2.2. $a_1' = \sup\{a_1, a\}$ and $a_k' = \sup\{a_n, a\}$;

2.3. $a_1' = \inf\{a_1, a\}$ and $a_k' = \inf\{a_n, a\}$.

Since the lattice of all ring topologies of a nilpotent ring is not modular (see [1]), then the question naturally arises of the validity of Theorem 2 for this lattice.

Theorem 3. [2] If R is a nilpotent ring and $a_1 < a_2 < \cdots < a_n$ is an unrefinable chain of ring topologies, then $k \le n$ for any chain $a_1' < a_2' < \cdots < a_k'$ of ring topologies such that $a_1 = a_1'$ and $a_n = a_k'$.

3 Notations and new results

Notations: *N is the set of all natural numbers;*

R *is the set of all matrices of the dimension* 3×3 *over the field of real numbers of the form*

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

R' *is the set of all matrices* of R for which $a_{13} = a_{23} = 0$*;*

R'' *is the set of all matrices of* R for which $a_{12} = a_{13} = 0$*;*

$R_i = R$, $R'_i = R'$ *and* $R_i'' = R''$ *for every natural number* i *;*

$\tilde{R} = \sum_{i=1}^{\infty} R_i$, $\tilde{R}' = \sum_{i=1}^{\infty} R_i'$ *and* $\tilde{R}'' = \sum_{i=1}^{\infty} R_i''$

$\tilde{\tilde{V}}_n = \{\tilde{g} \in \tilde{\tilde{R}} \mid pr_i(\tilde{g}) = 0 \ if \ i \ge n\}$*;*

Remark 1. *It is easy to see that* R *with the usual operation of matrix is a ring and* $R^3 = 0$*, and* $(R')^2 = (R'')^2 = 0$ *.*

Proposition 1. For the ring $\tilde{R}$ the following statements are true:

1.1. *The collection* $\boldsymbol{B}'=\{\tilde{V}_i \bigcap \tilde{R}' \mid i \in N\}$ *is a basis of neighborhoods of zero for a ring topology* $\tilde{\tau}'$ *on the ring* $\tilde{R}$ *;*

1.2. *The collection* $\boldsymbol{B}''=\{\tilde{V}_i \bigcap R'' \mid i \in N\}$ *is a basis of neighborhoods of zero for a ring topology* $\tilde{\tau}''$ *on the ring* $\tilde{R}$ *;*

Theorem 4. *Let* $\tilde{\tau}'$ *and* $\tilde{\tau}''$ *be ring topologies on the ring* $\tilde{R}_k$ *defined in Proposition 1,* $\tilde{\tau}_d$ *is the discrete topology on the ring* $\tilde{R}$ *and* τ^* *is a coatom in the lattice of all ring topologies on the ring* $\tilde{R}$ *such that* $\tau^* \geq \tilde{\tau}'$. *Then between the topologies* $\inf\{\tilde{\tau}_d, \tilde{\tau}''\}$ *and* $\inf\{\tau^*, \tilde{\tau}''\}$ *there exists a chain of ring topologies on the ring* $\tilde{R}$ *which is infinitely decreasing and infinitely increasing.*

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The semireflexive subcategories

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Abstract

A categorical definition of the notion of semireflexive subcategory is formulated and some its properties are proved.

Key words: semireflexive subcategory, reflective subcategory, local convex topological Hausdorff vector spaces.

In the category $\mathcal{C}_2\mathcal{V}$ of the local convex topological Hausdorff vector spaces we use the following notations: $\mathbb{R}$ - the class of all nonzero reflective subcategories of category $\mathcal{C}_2\mathcal{V}$; $\mathcal{S}$ - the subcategory of spaces with weak topology; Π - the subcategory of complete spaces with a weak topology; if $\mathcal{B}$ is a class of monomorphisms, and $\mathcal{A}$ - a subcategory, then $\mathcal{S}_{\mathcal{B}}(\mathcal{A})$ is the subcategory of $\mathcal{B}$-subobjects of objects from $\mathcal{A}$. The subcategory $\mathcal{L}$ is called c-reflexive, if $\mathcal{S} \subset \mathcal{R}$ and the functor $l : \mathcal{C}_2\mathcal{V} \to \mathcal{L}$ is left exact (see [2]).

Let $\mathcal{L}, \mathcal{R} \in \mathbb{R}$. Then $\lambda_{\mathcal{R}}(\mathcal{L})$ is the full subcategory of all Z objects with the property: for any object $A \in |\mathcal{L}|$ any morphism $f : A \to Z$ extends through $\mathcal{R}$-replica r^A of object A: $f = g \cdot r^A$ for an g. It is known that $\lambda_{\mathcal{R}}(\mathcal{L}) \in \mathbb{R}$.

Definition 1. *Let $\mathcal{A}$ be a subcategory and $\mathcal{L}$ a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$. The object X is called $(\mathcal{L}, \mathcal{A})$-semireflexive, if his $\mathcal{L}$-replica belongs to subcategory $\mathcal{A}$. The full subcategory of all $(\mathcal{L}, \mathcal{A})$-semireflexive objects is called the semireflexive product of subcategories $\mathcal{L}$ and $\mathcal{A}$ and is denote by $\mathcal{R} = \mathcal{L} \star_{sr} \mathcal{A}$.*

We mention the following properties of the semireflexive product (see [3]).

Proposition 1. *The semireflexive product $\mathcal{L} \star_{sr} \mathcal{A}$ is closed relative to $(\varepsilon\mathcal{L})$-subobjects and $(\varepsilon\mathcal{L})$-factorobjects.*

Proof. Let $\mathcal{R} = \mathcal{L} \star_{sr} \mathcal{A}$, $A \in |\mathcal{R}|$, $b : X \to A \in |\varepsilon\mathcal{L}|$, and $l^X : X \to lX$ $\mathcal{L}$-replica of X. Because $b \in \varepsilon\mathcal{L}$, we have

$$l^X = f \cdot b \tag{1}$$

for an f and f is $\mathcal{L}$-replica of A. So $lX \in |\mathcal{A}|$, and $X \in |\mathcal{R}|$.

Let $\mathcal{R}$ be closed relative to $(\varepsilon\mathcal{L})$-factorobjects. Let $A \in |\mathcal{R}|$, $t : A \to Y \in |\varepsilon\mathcal{L}|$, and $l^A : A \to lA$ $\mathcal{L}$-replica of A. Then

$$l^A = g \cdot t \tag{2}$$

for an g which is $\mathcal{L}$-replica of Y. So $Y \in |\mathcal{R}|$.

Proposition 2. *Let $\mathcal{L} \in \mathbb{R}$ and $\mathcal{A}$ be a subcategory of category $\mathcal{C}_2\mathcal{V}$. Then $\mathcal{L} \star_{sr} \mathcal{A} = \mathcal{S}_{\varepsilon\mathcal{L}}(\mathcal{L} \cap \mathcal{A})$.*

Proof. Let $X \in |\mathcal{L} \star_{sr} \mathcal{A}|$, and $l^X : X \to lX$ $\mathcal{L}$-replica of X. Then $lX \in |\mathcal{A}|$, or $lX \in |\mathcal{L} \cap \mathcal{A}|$, and $l^X \in \varepsilon\mathcal{L}$. So $X \in \mathcal{S}_{\varepsilon\mathcal{L}}(\mathcal{L} \cap \mathcal{A})$.

Be it now $X \in |\mathcal{S}_{\varepsilon\mathcal{L}}(\mathcal{L} \cap \mathcal{A})|$. Then there exist an object $Z \in |\mathcal{L} \cap \mathcal{A}|$ and a morphism $b : X \to Z \in |\varepsilon\mathcal{L}|$. It is clear that b is $\mathcal{L}$-replica of X and $lX = Z \in |\mathcal{A}|$. So $X \in |\mathcal{L} \star_{sr} \mathcal{A}|$.

Corollary 1 [3]. *Let $\mathcal{L}$ be c-reflective subcategory, and $\mathcal{A}$ a subcategory of category $\mathcal{C}_2\mathcal{V}$. Then $\mathcal{L} \star_{sr} \mathcal{A}$ is a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$.*

Proof. Indeed, $((\varepsilon\mathcal{L})^\top, \varepsilon\mathcal{L})$ is a structure of left factorization, and $\mathcal{L} \cap \mathcal{A}$ is a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$. If $t^X : X \to tX$ is $(\mathcal{L} \cap \mathcal{A})$-replica of X, and

$$t^X = i^X \cdot p^X \tag{1}$$

is $((\varepsilon\mathcal{L})^\top, \varepsilon\mathcal{L})$-factorization of t^X, then p^X is $(\mathcal{L} \star_{sr} \mathcal{A})$-replica of X.

Definition 2. *Let $\mathcal{L}$ and $\mathcal{R}$ be two reflective subcategories of category $\mathcal{C}_2\mathcal{V}$. $\mathcal{R}$ is called $\mathcal{L}$-semireflexive, if it is closed relative to $(\varepsilon\mathcal{L})$-subobjects and $(\varepsilon\mathcal{L})$-factorobjects. The class of all $\mathcal{L}$-semireflexive subcategories we denote by $\mathbb{R}^s_f(\varepsilon\mathcal{L})$.*

Examples. 1. Let $\mathcal{L}_1 \subset \mathcal{L}_2$. Then $\varepsilon\mathcal{L}_2 \subset \varepsilon\mathcal{L}_1$ and $\mathbb{R}^s_f(\varepsilon\mathcal{L}_1) \subset \mathbb{R}^s_f(\varepsilon\mathcal{L}_2)$.

2. $\mathbb{R}^s_f(\varepsilon\mathcal{C}_2\mathcal{V}) = \mathbb{R}$.

Theorem 1. *Let $\mathcal{R} \in \mathbb{R}^s_f(\varepsilon\mathcal{L})$ and $\mathcal{H} \in \mathbb{R}$. Then the following conditions are equivalent:*

1. $\mathcal{L} \star_{sr} \mathcal{H} = \mathcal{R}$.

2. $\mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{H}$.

Proof. $1 \Rightarrow 2$. $\mathcal{L} \cap \mathcal{R} \subset \mathcal{L} \cap \mathcal{H}$. Indeed, let $A \in |\mathcal{L} \cap \mathcal{R}|$. Then $lA = A \in |\mathcal{R}|$. So $lA \in |\mathcal{H}|$ or $A \in |\mathcal{H}|$, i.e. $A \in |\mathcal{L} \cap \mathcal{H}|$.

$\mathcal{L} \cap \mathcal{H} \subset \mathcal{L} \cap \mathcal{R}$. Let $A \in |\mathcal{L} \cap \mathcal{H}|$. Then $A \in |\mathcal{R}|$, or $A \in |\mathcal{L} \cap \mathcal{R}|$.

$2 \Rightarrow 1$. $\mathcal{L} \star_{sr} \mathcal{H} \subset \mathcal{R}$. Let $A \in |\mathcal{L} \star_{sr} \mathcal{H}|$ and $l^A : A \to lA$ is $\mathcal{L}$-replica to A. Then $lA \in |\mathcal{L} \cap \mathcal{H}| = |\mathcal{L} \cap \mathcal{R}|$. Also $lA \in |\mathcal{R}|$ and because $\mathcal{R} \in \mathbb{R}^s_f(\varepsilon\mathcal{L})$ it follows that $A \in |\mathcal{R}|$.

$\mathcal{R} \subset \mathcal{L} \star_{sr} \mathcal{H}$. Let $A \in |\mathcal{R}|$ and $l^A : A \to lA$ is $\mathcal{L}$-replica of A. Because $\mathcal{R} \in \mathbb{R}^s_f(\varepsilon\mathcal{L})$ it follows that $lA \in |\mathcal{R}|$. Also $lA \in |\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{H}|$, ie $A \subset \mathcal{L} \star_{sr} \mathcal{H}$.

Corollary 2. *Let $\mathcal{R} \in \mathbb{R}^s_f(\varepsilon\mathcal{L})$. The following statements are true:*

1. $\mathcal{R} = \mathcal{L} \star_{sr} (\mathcal{L} \cap \mathcal{R})$.

2. Let $\mathcal{R} = \mathcal{L} \star_{sr} \mathcal{H}$. Then $\mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{H}$.

3. Let $\mathcal{R} = \mathcal{L} \star_{sr} \mathcal{H}$, $\mathcal{T} \in \mathbb{R}$ and $\mathcal{L} \cap \mathcal{R} \subset \mathcal{T} \subset \mathcal{H}$ be. Then $\mathcal{R} = \mathcal{L} \star_{sr} \mathcal{T}$.

Theorem 2. *Let $\mathcal{R} \in \mathbb{R}^s_f(\varepsilon\mathcal{L})$. The following statements are true:*

1. For any $\mathcal{T} \in \mathbb{R}$ and $\mathcal{L} \subset \mathcal{T}$ we have $\mathcal{R} = \mathcal{T} \star_{sr} \mathcal{R}$.

2. Let $\mathcal{H} \in \mathbb{R}$ and $\mathcal{L} \cap \mathcal{R} \subset \mathcal{H} \subset \lambda_{\mathcal{R}}(\mathcal{L})$. Then $\mathcal{R} = \mathcal{L} \star_{sr} \lambda_{\mathcal{R}}(\mathcal{H})$.

3. For the semireflexive product both left and right factors are not uniquely determined.

Proof. 1. If $\mathcal{L} \subset \mathcal{T}$, then $\varepsilon\mathcal{T} \subset \varepsilon\mathcal{L}$.

2. We will check that $\mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{H}$. The inclusion $\mathcal{L} \cap \mathcal{R} \subset \mathcal{L} \cap \mathcal{H}$ is evident.

$\mathcal{L} \cap \mathcal{H} \subset \mathcal{L} \cap \mathcal{R}$. Let $A \in |\mathcal{L} \cap \mathcal{H}|$. Then $A \in |\lambda_{\mathcal{R}}(\mathcal{L})|$.

So the identical morphism $1 : A \to A$ extends through r^A. Whence it follows that $A \in |\mathcal{R}|$, or $A \in |\mathcal{L} \cap \mathcal{R}|$.

3. It results from p.1 and p.2.

Using the above notations we will examine some examples.

1. For the subcategories of semireflexive spaces $s\mathcal{R}$ we have: $s\mathcal{R} = \mathcal{S} \star_{sr} q\Gamma_0$, where $q\Gamma_0$ is the subcategory of quasicomplete spaces ([4]).

2. In the work [1] are defined the semireflexive inductive spaces and is proved that $i\mathcal{R} = \mathcal{S}h \star_{sr} \Gamma_0$, where $\mathcal{S}h$ is the subcategory of Schwatz spaces, and Γ_0 - the subcategory of complete spaces.

3. In the papers [4, 5] are defined the locally complete spaces (the subcategory $l\Gamma_0$). We prove that $l\Gamma_0 \in \mathcal{R} \in \mathbb{R}^s_f(\varepsilon\mathcal{S})$ ([4] Affirmation 2.5).

4. $\mathbb{R}^s_f(\varepsilon\mathcal{S}) \cap \mathbb{R}(\mathcal{E}_u) = \{\mathcal{C}_2\mathcal{V}\}$, where $\mathbb{R}(\mathcal{E}_u)$ is the class of $\mathcal{E}_u$-reflective subcategories, i.e. the needles $\mathcal{L} \in \mathbb{R}$, for which $\mathcal{S} \subset \mathcal{L}$.

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On some groupoids of small order

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Abstract

We count the number of groupoids of orders 2 and 3 with some Bol-Moufang type identities.

Keywords: groupoid, left (right) semi-medial identity, Manin identity, associativity.

1 Introduction

A binary groupoid $(G, \cdot)$ is a non-empty set G together with a binary operation "$\cdot$". This definition is very general, therefore usually groupoids with some identities are studied. For example, groupoids with associative law (semi-groups) are investigated.

Here we continue the study of groupoids with some Bol-Moufang type identities [5, 1, 8]. A groupoid $(Q, *)$ is called a quasigroup, if the following conditions are true [1]: $(\forall u, v \in Q)(\exists!\, x, y \in Q)(u * x = v \,\&\, y * u = v)$.

For quasigroups and semigroups the following natural problems are studied: how many quasigroups and semigroups of small order does exist? The number of semigroups of orders up to 8 is given in [7]; the number of quasigroups of orders up to 11 is given in [6, 10].

2 Some results

An original algorithm is elaborated and the corresponding program is written for generating of groupoids of small (2 and 3) orders with some Bol-Moufang identities, which are well known in quasigroup theory.

The studied below identities have the property that any of them defines a commutative Moufang loop [2, 1, 6, 8] in the class of loops (left (right) semimedial identity, Cote identity and its dual identity, Manin identity and its dual identity) or in the class of quasigroups (identity $(xy * x)z = (y * xz)x$ and its dual identity).

To verify the correctness of the written program the number of semigroups of order 3 was counted. The obtained result coincide with the well known, namely, there exist 113 semigroups of order 3.

2.1 Groupoids with left semi-medial identity

Left semi-medial identity in a groupoid $(Q, *)$ has the following form: $xx{*}yz = xy{*}xz$. Bruck [2, 1, 8] used this identity to define commutative Moufang loops in the class of loops.

There exist 10 left semi-medial groupoids of order 2. There exist 7 non-isomorphic left semi-medial groupoids of order 2, five of them are semigroups [10].

$$\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 1 \end{array} \quad \begin{array}{c|cc} \star & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 2 \end{array} \quad \begin{array}{c|cc} \circ & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 2 \end{array} \quad \begin{array}{c|cc} \cdot & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 1 & 2 \end{array}$$

$$\begin{array}{c|cc} \diamond & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 1 \end{array} \quad \begin{array}{c|cc} \odot & 1 & 2 \\ \hline 1 & 2 & 1 \\ 2 & 2 & 1 \end{array} \quad \begin{array}{c|cc} \bullet & 1 & 2 \\ \hline 1 & 2 & 2 \\ 2 & 1 & 1 \end{array}$$

There exist 399 left semi-medial groupoids of order 3.

Similar results are true for groupoids with right semi-medial identity $xy * zz = xz * yz$. It is clear that the identities of left and right semi-mediality are dual. In other words, they are (12)-parastrophes of each other [1, 8].

It is clear that groupoids with dual identities have similar properties, including the number of groupoids of a fixed order.

2.2 Groupoids with Cote identity

The identity $x(xy * z) = (z * xx)y$ is discovered in [3]. Here we call it the Cote identity.

There exist 6 groupoids of order 2 with Cote identity. There exist 3 non-isomorphic in pairs groupoids of order 2 with Cote identity.

There exist 99 groupoids of order 3 with Cote identity.

Similar results are true for groupoids with the identity $(z * yx)x = y(xx * z)$, which is the (12)-parastrophe of Cote identity.

2.3 Groupoids with Manin identity

The identity $x(y * xz) = (xx * y)z$ we call Manin identity [4]. The following identity is dual to Manin identity: $(zx * y)x = z(y * xx)$.

There exist 10 groupoids of order 2 with Manin identity. There exist 7 non-isomorphic in pairs groupoids of order 2 with Manin identity.

There exist 167 groupoids of order 3 with Manin identity.

2.4 Groupoids with identity $(xy * x)z = (y * xz)x$

Some properties of identity $(xy*x)z = (y*xz)x$ are given in [9, 8]. The following identity is dual to the identity (2.4): $z(x * yx) = x(zx * y)$.

There exist 6 groupoids of order 2 with identity (2.4). There exist 3 non-isomorphic in pairs groupoids of order 2 with (2.4) identity. Any of these groupoids is a semigroup.

There exist 117 groupoids of order 3 with identity (2.4).

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Subtractive Topological Quasigroups

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Abstract

In this paper we extend some results of the theory of topological groups to the class of subtractive topological quasigroups with (n, m)-identities.

Keywords: subtractive topological quasigroups, groupoid with (n, m)-identities.

1 Introduction

In this paper we study a special class of topological groupoids with a division, namely, the class of subtractive topological quasigroups. We proved that, if P is an open compact set of a subtractive topological quasigroup G, then P contains an open compact subtractive subquasigroup Q. This result was obtained for topological groups by L.Pontrjagin (see [1]). The established results are related to the results of M. Choban and L. Chiriac in [3] and to the research papers [4,5,6].

2 Basic notions

A non-empty set G is said to be a *groupoid* relatively to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G, there is a unique element $ab \in G$.

A groupoid $(G, \cdot)$ is called a *quasigroup* if, for every $a, b \in G$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions (see [2]).

A groupoid $(G, \cdot)$ is called *subtractive* if it satisfies the laws $b \cdot (b \cdot a) = a$ and $a \cdot (b \cdot c) = c \cdot (b \cdot a)$, for all $a, b, c \in G$.

We consider a groupoid $(G,+)$. For every two elements $a,b \in (G,+)$, denote

$$\begin{aligned} 1\,(a,b,+) &= (a,b,+)\,1 = a+b, \\ n\,(a,b,+) &= a+(n-1)\,(a,b,+)\,, \\ (a,b,+)\,n &= (a,b,+)\,(n-1)+b, \end{aligned}$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a,b)$ and $(a,b)n$ instead of $n\,(a,b,+)$ and $(a,b,+)\,n$.

Let $(G,+)$ be a groupoid, $n \geq 1$ and $m \geq 1$.

The element e of a groupoid $(G,+)$ is called an (n,m)-zero of G if $e+e=e$ and $n\,(e,x)=(x,e)\,m=x$, for every $x \in G$.

If $e+e=e$ and $n\,(e,x)=x$, for every $x \in G$, then e is called an (n,∞)-zero. If $e+e=e$ and $(x,e)\,m=x$, for every $x \in G$, then e is called an (∞,m)-zero.

It is clear that $e \in G$ is an (n,m)-zero, if it is an (n,∞)-zero and an (∞,m)-zero.

Remark. *In the multiplicative groupoid $(G,\cdot)$ the element e is called an (n,m)-**identity**. The notion of the (n,m)-**identity** was introduced in [3].*

3 Main results

Theorem 1. *Let $(G,\cdot)$ be a multiplicative groupoid, $e \in G$ and the following statements hold:*

1. *$xe = x$ for every $x \in G$;*

2. *$x^2 = x \cdot x = e$ for every $x \in G$;*

3. *$x \cdot (y \cdot z) = z \cdot (y \cdot x)$ for all $x,y,z \in G$;*

4. *For every $a, b \in G$ there exists a unique point $y \in G$ such that $ya = b$.*

   Then e is a $(2,1)$-identity in G.

Example. Let $(G, +)$ be a commutative additive group with a zero 0. Consider a new binary operation $x \cdot y = x - y$. Then $(G, \cdot)$ is a subtractive quasigroup with a $(2,1)$-identity 0.

Theorem 2. *Let $(G, \cdot)$ be a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. *$xe = x$, for every $x \in G$;*

2. *$b \cdot (b \cdot a) = a$, for all $a, b \in G$;*

3. *$a \cdot (b \cdot c) = c \cdot (b \cdot a)$, for all $a, b, c \in G$;*

4. *If $xa = ya$, then $x = y$.*

   Then G is a subtractive quasigroup with a $(2,1)$-identity e.

Corollary 3. *Let $(G, \cdot)$ be a subtractive quasigroup, $e \in G$ and $xe = x$ for every $x \in G$. Then e is a $(2,1)$-identity.*

Proposition 4. *Let P be a subset of a topological subtractive quasigroup G and $e \in P$. If $P_1 = P \bigcap eP$, then:*

1. *$eP_1 = P_1$*
2. *If P is open, then P_1 is open too.*
3. *If P is closed, then P_1 is closed too.*
4. *If P is compact, then P_1 is compact too.*

Proposition 5. *Let G be a subtractive quasigroup. Then the mapping $f : G \to G$, where $f(x) = ex$, is an involutary automorphism, i.e. $f = f^{-1}$ and $f(x \cdot y) = f(x) \cdot f(y)$, for every $x, y \in G$.*

Theorem 6. *Let G be a subtractive topological quasigroup. If P is an open compact subset, such that $e \in P$, then P contains an open compact subtractive quaisgroup Q of G.*

Theorem 7. *Let G be a subtractive topological quasigroup with $(2, 1)$-identity e. If P is an open compact subset, such that $e \in P$, then P contains an open compact subtractive quaisgroup Q with a $(2, 1)$-identity e of G.*

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Algorithm for constructing finite idempotent cyclic semirings with commutative addition

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Abstract

The paper deals with finite idempotent cyclic semirings with commutative addition. The author establishes a connection between idempotent cyclic semirings with commutative addition and ideals of nonnegative integers. An algorithm for constructing these semirings is presented .

Keywords: finite algebra, semiring, idempotent, integers, algorithm.

1 Introduction

In this paper we report the algorithm for constructing additive finite idempotent cyclic semirings by ideal of nonnegative integers.

Studies of finite cyclic semirings with commutative addition were started in 2010 by A. S. Bestuzhev and E. M. Vechtomov [3,4]. So, in [4] A. S. Bestuzhev describes the structure of finite commutative idempotent cyclic semirings that can be imagined as an upper semilattice with the width $m \leq 3$. In theorems 3–5 of [5] E. M. Vechtomov and I. V. Lubyagina describes the structure of finite idempotent cyclic semirings with noncommutative addition ,,modulo" finite idempotent cyclic semirings with commutative addition.

2 A representation of fic-semirings through ideals of nonnegative integers

Definition 1. *A semiring is a set S equipped with associative binary operations addition $(+)$ and multiplication $(\cdot)$, such that multiplication left and right distributes over addition.*

A semiring S is called idempotent if $s + s = s$ for all $s \in S$ and it is called commutative if both operations are commutative. A semiring S with a neutral element 0 of $\langle S, +\rangle$, satisfying the property of multiplicativity (i. e., $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$), is called a semiring with zero. A semiring S with a neutral element 1 of $\langle S, \cdot\rangle$ is called a semiring with identity.

Definition 2. *A semiring S with identity 1 is called cyclic if there exists a generating element $a \neq 1$ such that every nonzero element in S is equal to a^n for some nonnegative integer n.*

If S is an commutative idempotent semiring, then the order $\leq$ such that $x \leq y \Leftrightarrow x + y = y$ turns $\langle S, +\rangle$ into an upper semilattice $[1, pp.\ 151-152]$.

If S is an commutative finite cyclic semiring S with generator $a \in S$ and order $n+1$, then $a^{n+1} = a^n$ [6].

In this paper we study a commutative finite idempotent cyclic semirings without zero, let us call it by fic-semiring.

Proposition 1 [6]. *A fic-semiring S with order $n+1$ is unique determined by tuple*

$$P = (p_0, p_1, \ldots, p_n) \in \{0, 1, \ldots, n\}^{n+1}, \qquad a^{p_i} = 1 + a^i. \tag{1}$$

Let us denote by $\mathbb{N}_n$ the semiring $\langle \{0, 1, \ldots, n\}, \oplus, \odot\ \rangle$ with operations $x \oplus y = \min\{x + y, n\}$ and $x \odot y = \min\{xy, n\}$ for all $x, y \in \{0, 1, \ldots, n\}$.

Theorem 1 [7]. *The following properties of set $I = \{p_i : i \in \mathbb{N}_n\}$ of all elements of tuple (1) are true:*

1) $I = \{i \in \mathbb{N}_n : p_i = i\}$;

2) $I = \{i \in \mathbb{N}_n : a^i \geqslant a^0\}$;

3) I is ideal of semiring $\mathbb{N}_n$.

The semiring $\mathbb{N}_n$ has properties like semiring of nonnegative integers. Therefore, $I = \left\{\bigoplus_{i=1}^{k} \alpha_i \odot g_i : \alpha_i \in \mathbb{N}_n\right\}$ for some $g_1, \ldots, g_k \in \mathbb{N}_n$. The set $G = \{g_1, \ldots, g_k\}$ is called basis of I. If equation $g_j = \bigoplus_{i=1}^{k} \alpha_i \odot g_i$ consequences $\alpha_j = 1$ and $\alpha_i = 0$ for all $i \neq j$, then the basis G is called reduced.

3 Algorithm for constructing fic-semiring

Let $I \subseteq \mathbb{N}_n$ is ideal with basis G and $I_j = \{i \oplus j : i \in \mathbb{N}_n\}$. If fic-semiring S is determined by tuple 1, then $p_j = \min(I \cap I_j)$.

Theorem 2 (A criteria of existence of fic-semirings). *A fic-semiring S with order $n+1$ there exists if and only if $I \cap I_j = I_{p_j}$ for all $j \in \mathbb{N}_n$.*

It follows next algorithm.

Algorithm. *Let us consider ideal J of nonnegative integers and its reduced basis $G = \{g_1, g_2, \ldots, g_k\}$. Let $J_i = \{j + i : j \in J\}$, $p_i = \min(J \cap J_i)$ for all nonnegative integer $i \leq n$, and let us denote*

$$m = \min G, \quad M = \max G, \quad N = \min \bigcup_{j=1}^{m-1} (J \cap J_j) \setminus J_{p_j}\big).$$

For each reduced basis G and each integer n, such that $M < n \leqslant N$ or $n = M = m$, exists unique finite commutative idempotent cyclic semiring with order $n+1$. This semiring is represented by tuple $P = (0, p_1, \ldots, p_n)$.

Using the Sage Math system for mathematical computations, we find commutative fic-semirings with small order (Table 1).

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Table 1. Number of commutative FIC-semirings

$\lvert G\rvert$	Order $n+1$ of fic-semiring													
	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	-	-	-	1	1	4	5	10	11	17	21	30	32	43
3	-	-	-	-	-	-	-	2	2	6	6	17	17	32
4	-	-	-	-	-	-	-	-	-	-	-	-	-	2
Total	1	2	3	5	6	10	12	20	22	33	38	59	62	91

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Preservation of radicals by generalizations of derivations

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Abstract

By results of Anderson and Slin'ko, derivations preserve the locally nilpotent and nil radicals of algebras over a field of characteristic 0. There is also a well known and elementary result that derivations preserve idempotent ideals. The radical results are extended to some (not all) rings, possible generalizations using generalizations of derivations are examined, the relevance of the result about idempotent ideals is pointed out and some comments about the Jacobson radical are included.

Keywords: Algebra, radical, derivation.

1 Introduction

Slin'ko [1] showed that if A is an algebra over a field of characteristic 0, then for every algebra derivation d of A we have $d(\mathcal{L}(A) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A) \subseteq \mathcal{N}(A)$, where $\mathcal{L}$, $\mathcal{N}$ are the locally nilpotent and nil radical respectively. Using similar techniques, Anderson [2] had earlier obtained such results for algebras with a chain condition. We generalize these results in two ways, by replacing algebras by rings and replacing derivations by other mappings or sequences of mappings. We are unable to do so for *all* rings and there remain some open questions relating to the action of these generalizations of derivations. Here are the generalizations we shall consider.

A *higher derivation* of a ring R is a sequence $(d_0, d_1, \dots, d_n, \dots)$ of additive endomorphisms of R such that $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$, $a, b \in R$ for every n.

If α and β are endomorphisms of R, then an (α, β)-*derivation* of A is an additive endomorphism d such that $d(ab) = d(a)\beta(b) + \alpha(a)d(b)$ for all $a, b \in R$, so that an ordinary derivation is an (id, id)-derivation.

2 Extension to rings

If a ring R is additively torsion-free, we can extend the multiplication of R to the *divisible hull* $D(R)$ of (the additive group of) R. Then $D(R)$ becomes an algebra over the field $\mathbb{Q}$ of rational numbers. Moreover, each endomorphism of R extends uniquely to an algebra endomorphism of $D(R)$, and each derivation of R extends uniquely to an algebra derivation. From these results it is fairly straightforward to find analogous extensions for higher derivations and (α, β)-derivations. We thence get a version of the Anderson-Slin'ko results for these rings.

Theorem 1. *If a ring R is additively torsion-free, then $d(\mathcal{L}(R)) \subseteq \mathcal{L}(R)$ and $d(\mathcal{N}(R)) \subseteq \mathcal{N}(R)$ for every derivation d of R.*

For torsion rings the situation is unclear. It is routine to show that a radical is preserved on torsion rings if and only if it is preserved on p-rings (i.e. rings whose additive groups are p-groups), but there are $\mathbb{Z}_p$-algebras for which derivations need not preserve $\mathcal{L}$ and $\mathcal{N}$, by a result of Krempa [3]. For mixed (neither torsion nor torsion-free) rings it is equally unclear.

In the next two sections we shall give results for torsion-free rings. Analogous results for algebras in characteristic 0 hold, but we shall not mention this explicitly. In fact our method of proof is to establish the algebra case first and then proceed to the ring case in imitation of the procedure of this section.

3 Higher derivations

If a higher derivation has its zeroth mapping equal to the identity mapping *id*, then all its other mappings are linear combinations of compositions of derivations. This result has been proved many times,

first by Heerema [4] and most recently in a very interesting way by Hazewinkel [5] and is the source for part of the proof of our next result.
Theorem 2. *If a ring R is additively torsion-free, then for every higher derivation $(d_0, d_1, \ldots, d_n, \ldots)$ of R in which d_0 is an automorphism we have $d_n(\mathcal{L}(R)) \subseteq \mathcal{L}(R)$ and $d_n(\mathcal{N}(R)) \subseteq \mathcal{N}(R)$ for all n.*

For commutative rings without restriction *all* mappings of all higher derivations preserve our two radicals, which coincide with the set of nilpotent elements, but in non-commutative rings higher derivations need not preserve the set of nilpotent elements.

4 (α, β)-derivations

The best result we have here is the following.
Theorem 3. *If α is an automorphism of a torsion-free ring R, then $d(\mathcal{L}(R)) \subseteq \mathcal{L}(R)$ and $d(\mathcal{N}(R)) \subseteq \mathcal{N}(R)$ for every (α, α)-derivation d of R.*

It is not known how (α, β)-derivations treat radicals when α and β are unequal automorphisms, but there are examples of non-preservation for non-automorphisms, equal or not.

5 The relevance of idempotent ideals

In an arbitrary ring, derivations take idempotent ideals into themselves. There are radical classes which consist entirely of idempotent rings, but these are not our concern. Even $\mathcal{L}, \mathcal{N}$, even the prime radical can take idempotent values. Consider the algebra over a field which has a basis $\{e_t : t \in (0, 1)\}$ (here we refer to the real open interval) and multiplication given by $e_t e_u = e_{t+u}$ if $t + u < 1$ and 0 otherwise. This is idempotent and coincides with its prime radical. It is therefore worth stating
Theorem 4. *Let R be any ring, $\mathcal{R}$ a radical class. If $\mathcal{R}(R)$ is idempotent, then it is preserved by derivations, (α, β)-derivations where α and β are automorphisms and by higher derivations for which the zeroth mapping is an automorphism.*

6 What about the Jacobson radical?

The behaviour of the Jacobson radical with respect to derivations is complicated. In the power series ring (algebra) $\mathbb{Q}[[X]]$ formal differentiation does not preserve it. But Anderson's results show that it is preserved by derivations in algebras (characteristic 0) with DCC on two-sided ideals, and of course the Jacobson radical can be idempotent. We should also mention the Singer-Wermer Theorem [6] as strengthened by others: If A is a complex Banach algebra then for any derivation d, the Jacobson radical contains $d(A)$.

Full details of our results will appear elsewhere.

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Generalized WIP-quasigroups

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Abstract

Properties of generalized WIP-quasigroups (OWIP-quasigroups) are studied. Connections between OWIP-quasigroups and Bol quasigroups are established.

Keywords: quasigroup, loop, WIP-quasigroup, OWIP-quasigroup, Bol quasigroup, isotopy, LIP-loop.

WIP-loops and WIP-quasigroups are classical objects of quasigroup theory. A quasigroup L(+) has the weak-inverse-property if $x+I(y+x) = Iy$ for all $x, y \in L$ and some permutation of the set L [1, 2, 3]. Nuclei, autotopies, universality of identities were studied in this quasigroup class.

Definition. *Quasigroup $K(\cdot)$ is called OWIP-quasigroup, if in $K(\cdot)$ the following equality*

$$x \cdot I(y \cdot \alpha x) = Iy, \ \forall x, y \in K \qquad (1)$$

is true, where I and α are some permutations of the set K.

The following quasigroup is OWIP-quasigroup but it is not a WIP-quasigroup. Here I=(0 1), α = (1 2).

*	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

Lemma 1. *In OWIP-quasigroup $K(\cdot)$ the following identity is true*

$$I^{-1}(xz) \cdot \alpha x = I^{-1}z, \ \forall x, z \in K \qquad (2)$$

Proof. From (1) we have

$I^{-1}(x\cdot I(y\cdot\alpha x))=y,\ I^{-1}(x\cdot I(y\cdot\alpha x))\cdot\alpha x=y\cdot\alpha x,\ I^{-1}(xz)\cdot\alpha x=I^{-1}z$, where $y\cdot\alpha x=I^{-1}z$.

□

Theorem 2. *$OWIP$- quasigroup $K(\cdot)$ is isotopic to LIP- loop $K(\circ)$, where*

$$x\circ y=R_a^{-1}x\cdot L_b^{-1}y,\ xy=R_ax\circ L_by, \tag{3}$$

if and only if in $K(\cdot)$ the following equality is true

$$b\cdot I\left(I^{-1}(by)\cdot x\right)=R_{e_b}^{-1}\left(b\cdot I\left(I^{-1}b\cdot x\right)\right)\cdot y, \tag{4}$$

where $be_b=b,\ R_{e_b}v=ve_b$.

Proof. Let $K(\circ)$be an LIP- loop, i.e., in $K(\circ)$ the following equality is true: $I_lx\circ(x\circ y)=y$. Then using (3) we have $R_a^{-1}I_lx\cdot L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)=y$. We apply I^{-1} to both parts of the last equality and have $I^{-1}\left(R_a^{-1}I_lx\cdot L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)\right)=I^{-1}y$. We multiply the last equality from the right by expression $\alpha R_a^{-1}I_lx$ and have $I^{-1}\left(R_a^{-1}I_lx\cdot L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)\right)\cdot\alpha R_a^{-1}I_lx=I^{-1}y\cdot\alpha R_a^{-1}I_lx$. From equality (2) we obtain $I^{-1}L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)=I^{-1}y\cdot\alpha R_a^{-1}I_lx$. Further we do the following substitutions $x\to R_ax,\ y\to L_by$ and we have $I^{-1}L_b^{-1}(xy)=I^{-1}L_by\cdot\alpha R_a^{-1}I_lR_ax$. If $y=e_b$, where $be_b=b$, then

$I^{-1}L_b^{-1}R_{e_b}x=L_{I^{-1}b}\alpha R_a^{-1}I_lR_ax,\ \alpha R_a^{-1}I_lR_ax=L_{I^{-1}b}^{-1}I^{-1}L_b^{-1}R_{e_b}x,$

$I^{-1}L_b^{-1}(xy)=I^{-1}L_by\cdot L_{I^{-1}b}^{-1}I^{-1}L_b^{-1}R_{e_b}x,$

$R_{e_b}^{-1}L_bIL_{I^{-1}b}x\cdot y=L_bI\left(I^{-1}L_by\cdot x\right),\ R_{e_b}^{-1}\left(b\cdot I\left(I^{-1}b\cdot x\right)\right)\cdot y=b\cdot I\left(I^{-1}(by)\cdot x\right)$

We obtain (4).

Converse. Let (4) is true. Then we have $L_bI\left(I^{-1}L_by\cdot x\right)=\varphi x\cdot y$, where $\varphi x=R_{e_b}^{-1}L_bIL_{I^{-1}b}x$. Further we can write

$I^{-1}L_by\cdot\varphi^{-1}x=I^{-1}L_b^{-1}(xy),\ I^{-1}y\cdot\varphi^{-1}R_a^{-1}x=I^{-1}L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right).$

Therefore the following equality is true:

$$I^{-1}\left(\alpha^{-1}\varphi^{-1}R_a^{-1}x\cdot L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)\right)\cdot\alpha\left(\alpha^{-1}\varphi^{-1}R_a^{-1}x\right)\overset{(2)}{=}I^{-1}L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)=I^{-1}y\cdot\varphi^{-1}R_a^{-1}x$$

$$I^{-1}\left(\alpha^{-1}\varphi^{-1}R_a^{-1}x\cdot L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)\right)=I^{-1}y,$$

$$R_a^{-1}\left(R_a\alpha^{-1}\varphi^{-1}R_a^{-1}x\right)\cdot L_b^{-1}\left(R_a^{-1}x\cdot L_b^{-1}y\right)=y,$$

$R_a\alpha^{-1}\varphi^{-1}R_a^{-1}x\circ(x\circ y)=y,\ I_lx\circ(x\circ y)=y$, where $I_l=R_a\alpha^{-1}\varphi^{-1}R_a^{-1}$. □

Suppose that any loop isotope $K(\circ)$ of $OWIP$- quasigroup $K(\cdot)$, is an LIP-loop. In this case in quasigroup $K(\cdot)$ the following identity is holds:

$$z\cdot I\left(I^{-1}(zy)\cdot x\right)=R_{e_z}^{-1}\left(z\cdot I\left(I^{-1}z\cdot x\right)\right)\cdot y,\ \forall x,y,z\in K \qquad (5)$$

where I is a permutation of the set $K,\ \ ze_z=z,\ R_{e_z}t=te_z$.

Lemma 3. *If in quasigroup* $K(\cdot)$ *the identity* (5) *is true, then in* $K(\cdot)$ *the following identity holds* $z\cdot I(y\cdot\alpha z)=Iy,\ \forall z,y\in K$, (6)
where α *is a map of the set* K *into itself. If* α *is a permutation, then* $K(\cdot)$ *is an* $OWIP$ *- quasigroup.*

Proof. Equation $I\left(I^{-1}c\cdot x\right)=e_c$, where $ce_c=c$ has unique solutions. Indeed, $x=\left(IL_{I^{-1}c}\right)^{-1}e_c=L_{I^{-1}c}^{-1}I^{-1}e_c=\alpha c$. If we substitute in (5) $x=L_{I^{-1}z}^{-1}I^{-1}e_z=\alpha z$, then $I\left(I^{-1}(zy)\cdot\alpha z\right)=y,\ z\cdot I\left(I^{-1}(zy)\cdot\alpha z\right)=zy,$ where $t=I^{-1}(zy)$. □

Theorem 4. *Any loop* $K(\circ)$ *which is isotopic to quasigroup* $K(\cdot)$ *with identity (5), is a left Bol loop [4].*

Proof. It is sufficient to prove this fact for the loop $K(\circ)$, where isotopy has the form $x\circ y=R_a^{-1}x\cdot L_b^{-1}y,\ xy=R_ax\circ L_by$. (7)

From (5) and (7) we have

$R_az\circ L_bI\left(R_aI^{-1}(R_az\circ L_by)\circ L_bx\right)=R_aR_{e_z}^{-1}\left(z\cdot I\left(I^{-1}z\cdot x\right)\right)\circ L_by$. If $L_by=e$, where e is unit of the loop $K(\circ)$, $e=ba$, then we have

$$R_az\circ L_bI\left(R_aI^{-1}(R_az\circ L_by)\circ L_bx\right)=\left(R_az\circ L_bI\left(R_aI^{-1}R_az\circ L_bx\right)\right)\circ L_by.$$

We make the following substitutions: $z \to R_a^{-1} z$, $y \to L_b^{-1} y$, $x \to L_b^{-1} x$ and have $z \circ L_b I\left(R_a I^{-1}(z \circ y) \circ x\right) = \left(z \circ L_b I\left(R_a I^{-1} z \circ x\right)\right) \circ y$. If $z = e$, then we obtain $L_b I\left(R_a I^{-1} y \circ x\right) = \varphi x \circ y$, where $\varphi x = L_b I\left(R_a I^{-1} e \circ x\right)$, φ is a permutation of the set K. Therefore we obtain the following identity: $z \circ (\varphi x \circ (z \circ y)) = (z \circ (\varphi x \circ z)) \circ y$ or $z \circ (x \circ (z \circ y)) = (z \circ (x \circ z)) \circ y$, $\forall x, y, z \in K$. The last means that $K(\circ)$ is a left Bol loop. □

Corollary 5. *If quasigroup* $K(\cdot)$ *with identity (5) is an* LIP*- quasigroup, then* $K(\cdot)$ *is left Bol quasigroup, i.e., in* $K(\cdot)$ *identity* $x(y \cdot xz) = R_{e_x}^{-1}(x \cdot yx) \cdot z$ *holds, where* $xe_x = x$, $R_{e_x} t = te_x$ *[5].*

Right analogs of presented results are also true.

Some results of this paper are published in [6].

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Stable homotopy types and representation theory

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Abstract

We use the technique of the representation theory to the problem of classification of stable homotopy types of polyhedra. In particular, we establish when this problem is of finite, tame or wild type in the sense of the representation theory. In finite and tame cases a complete description of atoms (indecomposable polyhedra) is obtained.

Keywords: polyhedron, stable homotopy type, bimodule problem.

We denote by $\mathbf{S}$ the *stable homotopy category of polyhedra*. Its objects are punctured polyhedra (finite CW-compexes) and the set of morphisms from X to Y is defined as

$$\mathbf{S}(X,Y) = \varinjlim_n [S^n X, S^n Y],$$

where, as usually, $[X,Y]$ denotes the set of homotopy classes of maps $X \to Y$ and $S^n X$ is the n-th itereated suspension of X. It is known [1] that $\mathbf{S}$ is an additive triangulated category, where exact triangles are those isomorphic to the *cone triangles*

$$X \xrightarrow{f} Y \to Cf \to SX$$

(Cf is the cone of the map f). We denote by $\mathbf{S}_n$ the full subcategory of $\mathbf{S}$ consisting of $(n-1)$-connected polyhedra of dimension at most $2n-1$. Up to suspension, they present in the category $\mathbf{S}$ polyhedra having cells in at most n subsequent dimensions. Moreover, according to the generalized Freudenthal theorem [1], polyhedra from $\mathbf{S}_n$ are isomophic in $\mathbf{S}$ if and only if they are homotopically equivalent. Let $\mathbf{T}_n$ be the full subcategory of $\mathbf{S}_n$ consisting of *torsion free polyedra*, i.e. such that all homology groups $H_i(X,\mathbb{Z})$ are torsion free. A polyhedron $X \in \mathbf{S}_n$ is called an *atom* if it is not a suspension of a polyhedron from $\mathbf{S}_k$ for $k < n$ and is not isomorphic (in the category $\mathbf{S}$) to a non-trivial wedge (or bouquet) $X' \vee X''$.

Fix an integer m such that $0 < m < n$ and set $\mathbf{A} = \mathbf{S}_{n-m}$, $\mathbf{B} = \mathbf{S}_m$. Given a polyhedron $X \in \mathbf{S}_n$, there is an exact triangle

$$S^{2m-1}A \to S^{n-m}B \to X \to S^{2m}A,$$

where $A \in \mathbf{A}$, $B \in \mathbf{B}$. Consider the $\mathbf{A}$-$\mathbf{B}$-bimodule $\mathbf{M}(A,B) = \mathbf{S}(S^{2m-1}A, S^{n-m}B)$. Following [2], we can consider the *category* $\mathbf{El}(\mathbf{M})$ *of elements* of this bimodule. Its objects are morphisms from $\mathbf{M}(A,B)$ ($A \in \mathbf{A}$, $B \in \mathbf{B}$) and a morphism $f \to g$, where $f \in \mathbf{M}(A,B)$, $g \in \mathbf{M}(A',B')$ is a pair (α,β), where $\alpha : A \to A'$, $\beta : B \to B'$ such that the diagram

$$\begin{array}{ccc} S^{2m-1}A & \xrightarrow{f} & S^mB \\ {\scriptstyle S^{2m-1}\alpha}\big\downarrow & & \big\downarrow{\scriptstyle S^m\beta} \\ S^{2m-1}A' & \xrightarrow{g} & S^mB'. \end{array} \tag{1}$$

is commutative.

Theorem 1. *Mapping f to Cf, we obtain an equivalence of the categories $\mathbf{El}(\mathbf{M})/I \simeq \mathbf{S}_n/J$, where I is the ideal of morphisms of the form* (1) *such that β factors through g and J is the ideal of morphisms which factors both through an object from $S^{2m}\mathbf{A}$ and from an object from $S^{n-m}\mathbf{B}$. Moreover, $J^2 = 0$, hence isomorphism classes in $\mathbf{S}_n$ and in $\mathbf{S}_n/J$ are the same.*

Theorem 2. *Let $\mathbf{A}' = \mathbf{T}_{n-m}$, $\mathbf{B}' = \mathbf{T}_m$ and $\mathbf{El}(\mathbf{M}')$ be the full subcategory of $\mathbf{El}(\mathbf{M})$ consisting of such morphisms $f \in \mathbf{S}(S^{2m-1}A, S^{n-m}B)$ that $A \in \mathbf{A}'$, $B \in \mathbf{B}'$ and $H^m(f,\mathbb{Z}) = 0$. The map $f \mapsto Cf$ induces an equivalence of the categories $\mathbf{El}(\mathbf{M}')/I' \simeq \mathbf{T}_n/J'$, where*

$I' = I \cap \mathbf{El}(\mathbf{M}')$, $J' = J \cap \mathbf{T}_n$. *Moreover, both* ${I'}^2 = 0$ *and* ${J'}^2 = 0$, *so this equivalence induces a one-to-one correspondence between isomorphism classes in* $\mathbf{El}(\mathbf{M}')$ *and* $\mathbf{T}_n$.

Using these results and the technoque of the representation theory (*matrix problems*), we describe atoms in the categories $\mathbf{S}_n$ for $n \le 4$ and in the categories $\mathbf{T}_n$ for $n \le 7$. This description can be found in [3, 4]. Here we only present its qualitative consequences.

Theorem 3. 1. *The atoms in the categories* $\mathbf{S}_n$ *with* $n \le 3$ *have at most* 4 *cells. There is finitely many configurations of such atoms.*

2. *The category* $\mathbf{S}_4$ *is* tame *in the sense that its atoms are paramtrized by several "discrete" parameters and at most one "continuous" parameter. Namely, discrete parameters define the configuration of an atom,while the continuous parameter, which appears for some configurations, is actually a degree of an irreducible polynomial over a field.*

Here "configuration" defines the dimensions of cells that occur in the polyhedron and the elements of homotopy groups that occur in glueing cells of bigger dimensions to those of smaller dimensions.

Theorem 4. 1. *The categories* $\mathbf{T}_n$ *with* $n \le 6$ *have finitely many atoms. Each of them has at most* 2 *atoms if* $n \le 4$, *at most* 4 *atoms if* $n = 5$ *and at most* 6 *atoms for* $n = 6$.

2. *The category* $\mathbf{T}_7$ *is tame.*

Note that atoms from $\mathbf{S}_4$ and $\mathbf{T}_7$ can have arbitrary number of cells.

Finally, we establish that these results are exhaustive in some sense.

Theorem 5. *The classification of polyhedra from* $\mathbf{S}_n$ *for* $n > 4$ *or in* $\mathbf{T}_n$ *for* $n > 7$ *is a* wild problem *in the sense that it contains the problem of classification of representations of all finitely generated algebras over a field.*

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Generalized Boolean Algebras as Single Composition Systems for Measure Theory

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Abstract

Considering top element of Boolean algebras *optional*, Stone extended their class to "generalized Boolean algebras" (GBAs), and initiated their research by methods of abstract algebra, but also stated that his treatment of these structures is not the "most natural". Attributing his dissatisfaction to the treatment of GBAs as "*double* composition systems", these are presented here as commutative monoids with invertible operations, though in contrast with groups, not uniquely invertible. This treatment as "*single* composition systems" turn GBAs into "algebras of measurable objects" suggesting their extensive use in measure theory.

Keywords: (generalized) Boolean algebra, Stone representation theorem, algebras of sets, measure theory

1 Introduction

In [1-3], Stone laid down foundations of a theory of Boolean algebras (BAs) in compliance with practices of what was during his times said to be "modern algebra" and now is called "abstract algebra". However, in [3] he remarked that "the most natural approach" to BAs should "*not* be based upon the material of the present paper". To find the source of his dissatisfaction one needs to identify the difference between the approach to these structures generally adopted at his time and today (since this approach did not change over time), and the general approach to structures specific to "abstract algebra".

The "classical" structures of abstract algebra are groups, rings, fields, modules, vector spaces and algebras, and in this list, starting with rings, these structures are abelian groups equipped with additional operations. Groups are also a generalization of abelian groups and, thus, abelian groups can be said to be structures central in abstract algebra. The BAs and the wider class of "generalized Boolean algebras" (GBAs) introduced in [3] are "non-classical" since they do *not* fit in this pattern. BAs are generally treated as "double composition systems", since conjunction and disjunction are regarded as two "composition operations" of same importance. One can be expressed through the other (and negation) due to De Morgan laws, but no reason was ever presented to prefer one of them to another. Stone presented in [3] an alternative axiomatics for Boolean algebras and introduced the GBAs, but also as "double composition systems".

In this paper, GBAs (in particular, BAs) are treated as special commutative monoids with disjunction as their fundamental operation, that is, as "single composition systems", whereas the conjunction is defined through the fundamental operation. GBAs have no negation, De Morgan's laws make no sense in GBAs, and conjunction cannot be expressed through other operations like in BAs. This shows that presenting GBAs as commutative monoids is not a trivial task.

There is yet another reason for this representation – one residing in the nature of mathematics as a science about measure (in [4], the measure is related to "foundations of physics") – in this paper the GBAs are viewed as *algebras of measurable objects* – a view supported by the fact that the main interesting examples of GBAs are the Lebesgue or Borel measurable sets of finite measure in n-space ([3]). Measure is a function defined on an algebra of sets in terms of union (even though, union may be of infinite families). Thus, one expects that the algebras on which the measure is defined are GBAs in new presentation.

Having adopted here this view, we also need a name (different from "Boolean ring") for the new presentation of GBAs. These can be treated as representing the relation between a whole and its parts, a relation called "mereosis" in [5], and this suggests using the term "mereologic algebra". But mereology is a wide domain with too many algebras proposed up to now, and other algebras may be also called this manner. Also, whereas "algebra of mesurables" might be a good name, the form "measurable(s) algebra" is confusive. Thus, the term "extension algebra" is used here for them, proceeding from idea that what we measure are "*extensions* in space or time" – length, volume, duration, etc.

2 Extension algebras

Two symbols, "+" for addition and "–" for "subtraction", make up the signature of this algebra, and its axioms are those of a commutative idempotent monoid as well as those below:

$$(a+b)-c=(a-c)+(b-c), \tag{1}$$

$$a-(b+c)=(a-b)-c, \tag{2}$$

$$a+(b-a)=a+b, \tag{3}$$

$$a+(a-b)=a, \tag{4}$$

$$(a-b)-c=(a-c)-(b-c), \tag{5}$$

$$a-(b-c)=(a-b)+(a-(a-c)). \tag{6}$$

Propositions. In extension algebra, the following identities hold:

$$a-a=b-b, \tag{7}$$

$$a-(a-b)=b-(b-a). \tag{8}$$

Definitions. In an extension algebra,

$$0 :=a-b, \tag{9}$$

$$a\cap b :=a-(a-b), \tag{10}$$

$$a\bigtriangleup b :=(a-b)+(b-a) \tag{11}$$

Theorem 1. For an extension algebra A, the algebra with A's support and with operations 0, $\bigtriangleup$, $\cap$ defined in (9), (10), (11) is a Boolean ring (not necessarily with a unit).

Theorem 2. The axioms of extension algebra are valid in a GBA, where $a - b$ is defined as the complement of $a \cap b$ in the principle ideal generated by a, which is a Boolean algebra.

The Boolean rings are known to be presentations of GBAs ([2]). From the two theorems above, one can infer, that the extension algebra is a third presentation of GBAs, and this presentation is useful for the measure theory.

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On the duality of the class of biorthomorphisms

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Abstract

In this study, we examine the adjoint biorthomorphisms on the order bidual of Archimedean unital f-algebras by using Arens products.

Keywords: f-algebra, Arens multiplication, order unit, orthomorphisms, biorthomorphisms, vector lattice.

1 Introduction

Let E be an Archimedean vector lattice. We say that an ordeer bounded bilinear map $T : E \times E \to E$ is a biorthomorphisms if its partial maps are orthomorphism on E. $Orth(E, E)$ denotes the set of all biorthomorphisms of E. The biorthomorphisms were introduced and studied by R. Yilmaz and K.Rowlands in [10]. The order structure of the set $Orth(E, E)$ of all biorthomorphisms was given by the following result in [10], [2]: Let E be an Archimedean vector lattice. Then, $Orth(E, E)$ is an Archimedean vector lattice. In particular, lattice operations are defined by $(T \vee S)(x, y) = T(x, y) \vee S(x, y)$ and $(T \wedge S)(x, y) = T(x, y) \wedge S(x, y)$ for all $T, S \in Orth(E, E)$ and $(x, y) \in E^+ \times E^+$. Let A be an Archimedean semiprime f-algebra. $Orth(A)$ denotes the set of all orthomorphisms,[11]. The mapping $m : Orth(A) \longrightarrow Orth(A, A)$ introduced in [2], [4], [6] and defined by $m(\pi)(xy) = \pi(xy) = \pi(x)y$ for all $\pi \in Orth(A)$ and $(x, y) \in A \times A$ is an injective lattice homomorphism.The question is that when $Orth(A)$ is a band or order ideal in $Orth(A, A)$.

2 Preliminary results

Let $T \in Orth(A, A)$. Then the mapping $T : A \times A \to A$, $T(x, y)$, is a bilinear separately orthomorphism. t is wellknown that if A is an f-algebra with unit, then the second order dual $A^{\sim\sim}$ is also a Dedekind complete f-algebra with unit, [7], [8]. $A^{\sim}$ denotes the order dual of A and $A^{\sim\sim}$ denotes the second order dual of A. By using Arens multiplication [3], [5], we establish the following bilinear mappings:

$$T^{\sim} : A^{\sim} \times A \longrightarrow A, T^{\sim}(f, x) = f(T(x, y)), (1)$$

$$T^{\sim\sim} : A^{\sim\sim} \times A^{\sim} \to A^{\sim}, T^{\sim\sim}(G, f)(x) = G(T^{\sim}(f, x)), (2)$$

$$T^{\sim\sim\sim} : A^{\sim\sim} \times A^{\sim\sim} \to A^{\sim\sim}, T^{\sim\sim\sim}(G, F)(f) == G(T^{\sim\sim}(F, f)), (3)$$

for all $x \in A, f \in A^{\sim}, F, G \in A^{\sim\sim}$. For unexplained notion and terminology we refer to the books [1, 9].

3 Embedding of orthomorphisms into biorthomorphisms

Lemma 2. *Let A be an Archimedean semiprime f-algebra with separating order dual $A^{\sim}$ and e_{α} be an approximate identity of $A^{\sim\sim}$ and $T \in Orth(A^{\sim\sim}, A^{\sim\sim})$. Then, $T \in m(Orth(A^{\sim\sim}))$ if and only if the net $(T(F, e_{\alpha})$ has a supremum in $A^{\sim\sim})$ for every $0 \leq F \in A^{\sim\sim}$.*

Theorem 3. *Let A be an Archimedean semiprime f-algebra with separating order dual $A^{\sim}$. Then, $m : Orth(A^{\sim\sim}) \to Orth(A^{\sim\sim}, A^{\sim\sim})$, $m(Orth(A^{\sim\sim})$ is an order ideal in $Orth(A^{\sim\sim}, A^{\sim\sim})$.*

Theorem 4. *Let A be an Archimedean semiprime f-algebra with separating order dual $A^{\sim}$ and (e_{α}) be an approximate identity of $A^{\sim\sim}$. Then, the following are equivalent:*

(i) *The net* $(T(e_\alpha, x))$ *has a supremum in* $A^{\sim\sim}$ *for all* $x \in A^{\sim\sim}$ *and* $T \in Orth(A^{\sim\sim}, A^{\sim\sim})$.

(ii) m *is a lattice homomorphism.*

(iii) $Orth(A^{\sim\sim}, A^{\sim\sim})$ *is an f-algebra such that* m *is an algebra isomorphism.*

(iv) $m(Orth(A^{\sim\sim}))$ *is a band in* $Orth(A^{\sim\sim}, A^{\sim\sim})$

4 Conclusion

In this paper we research the vector space of all orthomorphims on the second order dual of an Archimedean semiprime f-algebra E is an order ideal in the set of all biorthomorphisms on the second order dual of E.

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Pseudo-automorphisms of middle Bol loops

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Abstract

The set of Moufang elements in a middle Bol loop is considered in the present work. We prove that every inner mapping of the Moufang part (which is a subloop) of a middle Bol loop $(Q,\cdot)$ extends to a right pseudo-automorphism of $(Q,\cdot)$.

Keywords: loop, multiplication group, inner mapping, middle Bol loop, pseudo-automorphism.

A grupoid $(Q,\cdot)$ is called a quasigroup if the equations $a\cdot x=b$ and $y\cdot a=b$ have unique solutions, for $\forall a,b\in Q$. A loop is a quasigroup with a neutral element. Two quasigroups $(Q,\cdot)$ and $(Q,*)$ are isotopic, if there exist $\alpha,\beta,\gamma\in S_Q$, such that $x*y=\gamma^{-1}(\alpha(x)\cdot\beta(y))$, $\forall x,y\in Q$. If (Q,A) is a quasigroup and $\sigma\in S_3$, then the operation ${}^{\sigma}A$, defined by the equivalence ${}^{\sigma}A(x_{\sigma(1)},x_{\sigma(2)})=x_{\sigma(3)}\Leftrightarrow A(x_1,x_2)=x_3$, is called a σ-parastrophe of the operation A. The product of an isotopy and a parastrophy, in any order, of a quasigroup $(Q,\cdot)$ is called an isostrophy of $(Q,\cdot)$.

A loop $(Q,\cdot)$ is called a middle Bol loop if it satisfies the identity: $x(yz\backslash x)=(x/z)(y\backslash x)$. It is proved in [4] that middle Bol loops are isostrophes of left (resp. right) Bol loops. Namely, a loop $(Q,\circ)$ is middle Bol if and only if there exists a right (left) Bol loop $(Q,\cdot)$, such that, $\forall x,y\in Q$:

$$x\circ y=y^{-1}\backslash x,\quad (resp.\ \ x\circ y=x/y^{-1}). \tag{1}$$

Let $(Q,\cdot)$ be a loop. We consider the sets:

$$M_l^{(\cdot)}=\{a\in Q\mid a(y\cdot az)=(ay\cdot a)z,\forall y,z\in Q\},$$

$$M_r^{(\cdot)} = \{a \in Q \mid (za \cdot y)a = z(a \cdot ya), \forall y, z \in Q\},$$

$$M^{(\cdot)} = \{a \in Q \mid ay \cdot za = a(yz \cdot a), \forall y, z \in Q\}.$$

Lemma 1. [3] *If $(Q,\cdot)$ is a middle Bol loop, then $M_l^{(\cdot)} = M_r^{(\cdot)} = M^{(\cdot)}$ and form a subbloop in $(Q,\cdot)$.*

Definition. *Let $(Q,\cdot)$ be a middle Bol loop. The subbloop $M^{(\cdot)}$ is called the Moufang part of $(Q,\cdot)$.*

Let $(Q,\cdot)$ be an arbitrary loop, $\varphi \in S_Q$ and $c \in Q$. Recall that:
a) φ is called a left (resp. right) pseudo-automorphism of $(Q,\cdot)$, with the companion c, if the equality

$$c \cdot \varphi(x \cdot y) = [c \cdot \varphi(x)] \cdot \varphi(y),$$

respectively,

$$\varphi(x \cdot y) \cdot c = \varphi(x) \cdot [\varphi(y) \cdot c],$$

holds, for every $x, y \in Q$.

Pseudo-automorphisms (left, right) have been introduced by Bruck [1] and were studied by many authors (see, for example, [1-3,5]). Bruck proved in [1] that every inner mapping of a Moufang loop is a pseudo-automorphism of this loop. Recall that a mapping α of the multiplication group $M(Q,\cdot) =< L_x^{(\cdot)}, R_y^{(\cdot)} | x, y \in Q >$ of a loop $(Q,\cdot)$ is called an inner mapping of $(Q,\cdot)$ if $\alpha(e) = e$, where e is the neutral element of this loop.

Theorem 1. *Let $(Q,\cdot)$ be a middle Bol loop. Each inner mapping of $M^{(\cdot)}$ extends to a pseudo-automorphism of $(Q,\cdot)$.*

Proof. Let $H =< L_x^{(\cdot)}, R_y^{(\cdot)} | x, y \in M^{(\cdot)} >$ be the multiplication group of the subloop $M^{(\cdot)}$, where $L_x^{(\cdot)}(z) = x \cdot z$ and $R_y^{(\cdot)}(z) = z \cdot y$, for all $z \in Q$. If $a \in M^{(\cdot)}$, then $L_a^{(\cdot)-1} = L_{a^{-1}}^{(\cdot)}$ and $R_a^{(\cdot)-1} = R_{a^{-1}}^{(\cdot)}$. Indeed, if $a \in M^{(\cdot)}$ then $ay \cdot za = a(yz \cdot a)$, for every $y, z \in Q$. Now, taking $z = a^{-1}$ in the last equality, we get:

$$a \cdot y = a \cdot (ya^{-1} \cdot a) \Rightarrow y = ya^{-1} \cdot a =$$
$$= R_a^{(\cdot)} R_{a^{-1}}^{(\cdot)}(y) \Rightarrow R_a^{(\cdot)-1}(y) = R_{a^{-1}}^{(\cdot)}(y),$$

$\forall y \in Q$. As $M^{(\cdot)} = M_l^{(\cdot)}$, for $a \in M^{(\cdot)}$ the equality $a(y \cdot az) = (ay \cdot a)z$ holds, for every $y, z \in Q$. Taking $y = a^{-1}$ in the last equality, we get:

$$a(a^{-1} \cdot az) = a \cdot z \Rightarrow a^{-1} \cdot az = z \Rightarrow L_{a^{-1}}^{(\cdot)} L_a^{(\cdot)}(z) = z,$$

$\forall z \in Q$, so $L_{a^{-1}}^{(\cdot)} = L_a^{(\cdot)-1}$.

If $U \in H$, then U can be expressed in the form $U = U_1 U_2 ... U_n$, where $U_i = R_{a_i}^{(\cdot)}$ or $U_i = L_{a_i}^{(\cdot)}$, for some $a_i \in M^{(\cdot)}$. Let $a \in M^{(\cdot)}$, then $ay \cdot za = a(yz \cdot a)$, for all $y, z \in Q$, so the triple

$$T_1 = (L_a^{(\cdot)}, R_a^{(\cdot)}, L_a^{(\cdot)} R_a^{(\cdot)})$$

is an autotopism of $(Q, \cdot)$. For each $a \in M^{(\cdot)} = M_r^{(\cdot)}$ we have $(za \cdot y)a = z(a \cdot ya)$, $\forall y, z \in Q$, so

$$(R_a^{(\cdot)-1}, L_a^{(\cdot)} R_a^{(\cdot)}, R_a^{(\cdot)})$$

is an autotopism of $(Q, \cdot)$ as well, hence the triple

$$T_2 = (R_a^{(\cdot)}, R_a^{(\cdot)-1} L_a^{(\cdot)-1}, R_a^{(\cdot)-1})$$

is an autotopism of $(Q, \cdot)$. As T_1 and T_2 are autotopisms of $(Q, \cdot)$ we get that, for all $U_i \in H$ there exists $V_i, W_i \in H$, such that

$$U_i(y) \cdot V_i(z) = W_i(y \cdot z),$$

$\forall y, z \in Q$. So, letting $V = V_1 V_2 ... V_n$ and $W = W_1 W_2 ... W_n$, we obtain: $W(y \cdot z) = W_1 W_2 ... W_n(y \cdot z) = W_1 W_2 ... W_{n-1}(U_n(y) \cdot V_n(z)) = ... = U_1 U_2 ... U_n(y) \cdot V_1 V_2 ... V_n(z) = U(y) \cdot V(z)$, for all $y, z \in Q$, so

$$U(y) \cdot V(z) = W(y \cdot z), \tag{2}$$

for all $y, z \in Q$. Let U be a inner mapping of $M^{(\cdot)}$, then $U(e) = e$, where e is the unit of $(Q, \cdot)$. Taking $y = e$ in (2) we obtain $V = W$, so $U(y) \cdot V(z) = V(y \cdot z)$, for all $y, z \in Q$. Taking $z = e$ in the last equality we have $U(y) \cdot V(e) = V(y)$, for all $y \in Q$. Denoting $V(e) = u$, we obtain that $V = R_u^{(\cdot)} U$, so the triple $T = (U, R_u^{(\cdot)} U, R_u^{(\cdot)} U)$ is an autotopism of $(Q, \cdot)$, which implies that U is a right pseudo-automorphism of the loop $(Q, \cdot)$, with the companion $u = V(e)$. $\square$

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About model completeness in the provability-intuitionistic logic and its extensions

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Abstract

A. Kuznetsov proposed the provability-intuitionistic logic which is an original extension of the propositional intuitionistic logic. A numerable family of closed classes of formulas of the provability-intuitionistic logic which are model pre-complete in this logic and in its non tabular extensions is presented in this paper.

Keywords: intuitionistic calculus, provability-intuitionistic logic, pseudo-boolean algebras, model complete system.

Provability-intuitionistic logic formulas are defined usually on the alphabet that consists of propositional variables $p, q, r, \ldots$, eventually with indices, symbols of operations

$$\&, \vee, \supset \neg, \Delta \tag{1}$$

and parentheses. The provability-intuitionistic propositional calculus I^{Δ} [1] is defined by the axioms of intuitionistic calculus [2], three Δ - axioms

$$(p \supset \Delta p), ((\Delta p \supset p) \supset p), ((p \supset q) \supset p) \supset (\Delta q \supset p)$$

and two inference rules: modus ponens and substitution rule.

We agree to define the logic I^{Δ} as the set of deductible formulas in the calculus I^{Δ}. In general, any set of formulas in the signature (1), which contains the axioms of the calculus I^{Δ} and is closed relative to the calculation rules for inference of this calculus, is called an extension of the provability-intuitionistic logic.

As an algebraic interpretation of the calculus I^{Δ} and its extensions served Δ - pseudo-boolean algebras, i. e. systems of the type $< E; \&, \vee, \supset \neg, \Delta >$, where $< E; \&, \vee, \supset \neg >$ is a pseudo-boolean algebra [3], and the relations $x \leq \Delta x$, $\Delta x \supset x = x$, $\Delta x \leq y \vee (y \supset x)$ take place. A formula is said to be *valid* on the algebra $\mathcal{A}$ if it is identically equal to the unit 1 of this algebra. The set of all formulas valid on the Δ-pseudo-boolean algebra $\mathcal{A}$ is an extension of the provability-intuitionistic logic. We call this logic as the logic of the algebra $\mathcal{A}$, and denote it by $L\mathcal{A}$. Let consider the logic of Δ-pseudo-boolean algebra $\mathcal{B} =< \{\tau_0, \tau_1, \tau_2, \ldots, 1\}; \&, \vee, \supset \neg, \Delta >$, where $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < 1$. Let L be an non-tabular extension of logic I^{Δ}, satisfying the condition $I^{\Delta} \subseteq L \subseteq L\mathcal{B}$. The set of non-tabular extensions of the logic I^{Δ}, that satisfy this condition, is of the continuous cardinal [4].

We say that a formula F is expressible in the logic L through the system of formulas Σ, if F can be obtained from variables and formulas of the system Σ by applying a finite number of times the weak rule of substitution, which allows the passage from two formulas to the result of substitution of one of them to other instead of all entries of some one of variables, and equivalent replacement rule in the logic L, which allows switching in L from one formula to a formula equivalent to her.

A formula $F(p_1, \ldots, p_n)$ of the logic L is called a *model* for boolean function $f(p_1, \ldots, p_n)$, if the identity $F(p_1, \ldots, p_n) = f(p_1, \ldots, p_n)$ is true on the set $\{0; 1\}$. The system Σ of formulas of the logic L is said to be *model complete* in L, if, for any boolean function, at least one of its model is expressible in L by Σ. The system Σ is called *model pre-complete* in L, if Σ is not model complete in L but for any formula F, which is not expressible in L by Σ, the system $\Sigma \bigcup \{F\}$ is model complete in L. The idea of a model completeness research belongs to A.V. Kuznetsov, a criterion for model completeness in general 3-valent logic was obtained by Iu. N. Tolstova [5].

We say that a formula $F(p_1, \ldots, p_n)$ preserves on the algebra $\mathcal{A}$

the predicate $R(x)$, if for any elements $\alpha_1, \ldots, \alpha_n$ of the algebra $\mathcal{A}$ the proposition $R(F[\alpha_1, \ldots, \alpha_n])$ is true each time when propositions $R(\alpha_1), \ldots, R(\alpha_n)$ are true. It is obvious, that a class of formulas that preserve on this algebra a predicate is closed relative to the expressibility in the logic of this algebra. Let denote by K_i $(i = 1, 2, \ldots)$ the class of formulas preserving the predicate $(x = 0) \vee (x = \tau_i)$ on the algebra $\mathcal{B}$.

Let consider the following application $f(p)$ of the algebra $\mathcal{B}$ into the logic I^{Δ}:

$$f(0) = (p\&\neg p), \ldots, f(\tau_i) = \delta^i(p\&\neg p), \ldots, f(1) = (p \supset p)(i = 1, 2, \ldots).$$

Lemma 1. *For any elements α and β of the algebra B, the following equivalences are deductible in I^{Δ}:*

$$(f(\alpha\&\beta) \sim (f(\alpha)\&f(\beta))), \tag{2}$$

$$(f(\alpha \vee \beta) \sim (f(\alpha) \vee f(\beta))), \tag{3}$$

$$(f(\alpha \supset \beta) \sim (f(\alpha) \supset f(\beta))), \tag{4}$$

$$(f(\Delta\alpha) \sim \Delta f(\alpha)), \tag{5}$$

$$(f(\neg\alpha) \sim \neg f(\alpha)). \tag{6}$$

Using deductions (2) - (6), it is demonstrated by induction that for the arbitrary formula $A(p_1, \ldots, p_n)$ and for any elements $\alpha_1, \ldots, \alpha_n$ of the algebra $\mathcal{B}$ the following statement is correct

$$\vdash (f(A[\alpha_1, \ldots, \alpha_n]) \sim A[p_1/f(\alpha_1), \ldots, p_n/f(\alpha_n)]).$$

Lemma 2. *Let f be the above-defined application of algebra $\mathcal{B}$ in the logic I^{Δ}. Then for any element $\tau_i \in \mathcal{B}(i = 1, 2, ...)$ the formulas $f(0)$ and $f(\tau_i)$ belong to the class K_i.*

Using these lemmas, is demonstrated the following theorem:

Theorem 1.*Let L be a non-tabular logic which satisfies the conditions $I^{\Delta} \subseteq L \subseteq L\mathcal{B}$. Classes $K_1, K_2, \ldots$ are distinct two by two relative to inclusion and they are model pre-complete in L.*

Theorem 2.*Let L be a non-tabular logic which satisfies the conditions $I^{\Delta} \subseteq L \subseteq L\mathcal{B}$. For logic L does not exist a criterion of model completeness traditionally formulated in terms of a finite number of model pre-complete classes.*

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On the orthogonal groupoids

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Abstract

The subject of this note is to investigate the question when the groupoid has an orthogonal complement and how many complements they are. If $|Q|=n$, then a finite groupoid $(Q,\cdot)$ has exactly $(n!)^n$ orthogonal complements.

Keywords: orthogonal groupoids, quasigroup,

The concept of orthogonality is introduced by the L. Euler in [1] for Latin squares (also called Greco-Latin squares). The extension of the notion of orthogonality from latin squares to arbitrary groupoid tables has been discussed by several authors, for exampl, A. Sade [3], S. K. Stein [2], V. Belousov [4].

A groupoid is a pair $(Q,*)$, where Q is a set and $''*''$ a binary operation on Q (a function from $Q\times Q$ to Q). A groupoid $(Q,*)$ is a quasigroup if an only if its Cayley table is a latin square. The number $|Q|$ is called the order of the corresponding latin square.

Definition 1. *Two groupoids* $(Q,\cdot),(Q,*)$ *are said to be orthogonal if, for any* $a,b\in G$ *the system of equations*

$$\begin{cases} x\cdot y=a \\ x*y=b \end{cases} \qquad (1)$$

has a unique solution in Q.

This Definition is equivalent to the fact that the mapping $(x,y)\mapsto(x\cdot y,x*y)$ is a permutation of $Q\times Q$. In this case the groupoid $(Q,*)$ is called an orthogonal complement for $(Q,\cdot)$.

Orthogonality is a symmetric property. It is rather easy to construct a pair $(Q,\cdot),(Q,*)$ of orthogonal groupoids on the set Q. We start from an arbitrary permutation φ of $Q\times Q$ and, for any $x,y\in Q$, put $x\cdot y=u,\ \ x*y=v$ if and only if $\varphi(x,y)=(u,v)$. So, there are $(|Q|^2)!$ pair of orthogonal groupoids defined on the set Q.

Proposition 1. *If a groupoid* $(Q,\cdot)$ *has an orthogonal complement than* $Q\cdot Q=Q$*, where* $Q\cdot Q=\{a\cdot b\mid a,b\in Q\}$.

Proof. Suppose the groupoid $(Q,\cdot)$ has an orthogonal complement $(Q,*)$, but $Q\cdot Q\subset Q$. Than there exists $a\in Q$, such that $a\notin Q\cdot Q$. So $x\cdot y\neq a$, for any $x,y\in Q$. Therefore, for any groupoid $(Q,*)$ and any element $b\in Q$, the system (1) has no solutions in Q. This means that the groupoid $(Q,\cdot)$ is not orthogonal to any groupoid $(Q,*)$.

The condition $Q\cdot Q=Q$ is not sufficient for the existence of an orthogonal complement for the groupoid $(Q,\cdot)$.

For $a\in Q$ we denote $S(a,\cdot)=\{(x,y)\in Q^2\mid x\cdot y=a\}$.

Proposition 2. *i) For all* $a,b\in Q$, $S(a,\cdot)\bigcap S(b,\cdot)=\varnothing\Leftrightarrow a\neq b$;

(ii) $\bigcup\limits_{a\in Q}S(a,\cdot)=Q\times Q$.

Theorem 1. *i) A finite groupoid* $(Q,\cdot)$ *has an orthogonal complement if and only if* $|S(a,\cdot)|=|Q|$*, for all* $a\in Q$*;*

ii) If $|Q|=n$*, and* $|S(a,\cdot)|=|Q|$*, for all* $a\in Q$*, then it has exactly* $(n!)^n$ *orthogonal complements.*

Proof. i) Let $|Q|=n$ and let $(Q,\cdot)$ has an orthogonal complement $(Q,*)$, but there exists $a\in Q$, such that $|S(a,\cdot)|\neq|Q|$. We can find $a\in Q$, such that $|S(a,\cdot)|>|Q|$. Than there exist $n+1$ diferent pairs $(x_1,y_1),(x_2,y_2),\ldots,(x_{n+1},y_{n+1})\in Q^2$, with $x_i\cdot y_i=a$, for $i\in\{1,2,\ldots,n+1\}$. Since $|\{x_i*y_i\mid i\in\{1,2,\ldots,n+1\}\}|\leq n=|Q|$, there exist at least two diferent pairs $(x_k,y_k),(x_j,y_j), 1\leq j,k\leq n+1$, such that $x_j \cdot y_j = x_k \cdot y_k = a$ and $x_j * y_j = x_k * y_k = b$, for some $b \in Q$. Therefore, the system (1) has two solutions, i.e. $(Q,\cdot)$ is not orthogonal to $(Q,*)$. So $|S(a,\cdot)|=|Q|$, for all $a \in Q$.

Conversely, suppose $|S(a,\cdot)|=|Q|$, for all $a \in Q$, and let $\alpha_a : S(a,\cdot) \to Q$ be an arbitrary one-to-one mapping. Obviously, $\alpha_a \neq \alpha_b$ if and only if $a \neq b$, since $S(a,\cdot) \cap S(b,\cdot) = \varnothing$. It is clear that $(x,y) \in S(x \cdot y,\cdot)$, and $\alpha_{x\cdot y} = \alpha_{w \cdot z}$ if and only if $x \cdot y = w \cdot z$. Using the system $\{\alpha_a \mid a \in Q\}$ we define the groupoid $(Q,*)$ as follows: for $x, y \in Q$, we put $x * y = \alpha_{x\cdot y}(x,y)$. We have $Q * Q = Q$ since $|S(x \cdot y,\cdot)|=|Q|$ and $\alpha_{x\cdot y} : S(x \cdot y,\cdot) \to Q$ is a one-to-one mapping. Groupoids $(Q,\cdot)$ and $(Q,*)$ are orthogonal. Indeed, let us fix any $a, b \in Q$. Since $Q \cdot Q = Q$ there are $x, y \in Q$ such as $x \cdot y = a$. Analogously, since $\alpha_{x\cdot y} : S(x \cdot y,\cdot) \to Q$ is a one-to-one mapping, there are unique $(x', y') \in S(x \cdot y,\cdot)$, such that $\alpha_{x\cdot y}(x', y') = b$. Obviously, $x' \cdot y' = x \cdot y$, $\alpha_{x\cdot y}(x', y') = \alpha_{x' \cdot y'}(x', y')$ and (x', y') is a unique solution of the system

$$\begin{cases} u \cdot v = a, \\ u * v = b. \end{cases}$$

Therefore, groupoids $(Q,\cdot)$ and $(Q,*)$ are orthogonal.

ii) Let $|Q|=n$ and $|S(a,\cdot)|=|Q|$, for all $a \in Q$. Let $\{\alpha_t : S(t,\cdot) \to Q \mid t \in Q\}$ and $\{\alpha'_t : S(t,\cdot) \to Q \mid t \in Q\}$ be two sets of one-to-one mappings. Define $x * y = \alpha_{x\cdot y}(x,y)$ and $x *' y = \alpha'_{x\cdot y}(x,y)$, for all $x, y \in Q$. Fix any $a \in Q$. If $\alpha_a \neq \alpha'_a$, then there exists $(c,d) \in S(a,\cdot)$, such that $\alpha_a(c,d) \neq \alpha'_a(c,d)$. So $x * y \neq x *' y$ and the groupoids $(Q,*)$ and $(Q,*')$ are different. Therefore, we have at least $(n!)^n$ different orthogonal complements of the groupoid $(Q,\cdot)$, since there are $(n!)^n$ different sets of one-to-one mappings $\{\alpha_t : S(t,\cdot) \to Q \mid t \in Q\}$.

Will show that any orthogonal complement $(Q,*)$ of a groupoid $(Q,\cdot)$ is obtained in the manner described above. Indeed, since $(Q,\cdot)$ and $(Q,*)$ are orthogonal, the system (1) has a unique solution for any pair $a,b \in Q$. From i) we have $|S(a,\cdot)|=|Q|$, for all $a \in Q$. For any $a \in Q$ define $\varphi_a : Q \to S(a,\cdot)$, namely, for every $b \in Q$ let us put $\varphi_a(b) = (x_{a,b}, y_{a,b})$ if and only if

$$\begin{cases} x_{a,b} \cdot y_{a,b} = a, \\ x_{a,b} * y_{a,b} = b. \end{cases}$$

The map φ_a is a bijection, and $\varphi_a \neq \varphi_c$ if and only if $a \neq c$, since $(Q,\cdot)$ and $(Q,*)$ are orthogonal. Now is easy to see that for $\{\alpha_a = \varphi_a^{-1} \mid a \in Q\}$ we have $x * y = \alpha_{x\cdot y}(x,y)$, for all $x, y \in Q$. Indeed,

$$\alpha_{x\cdot y}(x,y) = b \Leftrightarrow \phi_{x\cdot y}^{-1}(x,y) = b \Leftrightarrow \phi_{x\cdot y}(b) = (x\cdot y) \Leftrightarrow$$
$$\Leftrightarrow x * y = b \Leftrightarrow x * y = \alpha_{x\cdot y}(x,y).$$

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On the left coquotient with respect to meet for pretorsions in modules

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Abstract

In [2] the operation of *left coquotient with respect to meet* for preradicals of R-Mod is defined. In the present short notice the particular case of *pretorsions* of R-Mod is considered. We prove that for pretorsions the studied operation coincides with the operation (called *right residual*) introduced and investigated by J.S.Golan ([1]) in the terms of preradical filters of R. For that it is necessary to show the concordance of the studied operation with the transition $r \rightsquigarrow \mathcal{E}_r$ from pretorsions of R-Mod to the preradical filters of the ring R.

Keywords: module, pretorsion, filter, left coquotient.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. By definition a pretorsion is a hereditary preradical. We denote by $\mathbb{PT}$ the set of all pretorsions of the category R-Mod. It is well known the description of pretorsions by preradical filters.

Definition. *The set of left ideals $\mathcal{E} \subseteq \mathbb{L}(_RR)$ is called* preradical filter *(left linear topology) if it satisfies the following conditions:*

(a_1) *If $I \in \mathcal{E}$ and $a \in R$, then $(I : a) = \{x \in R \mid xa \in I\} \in \mathcal{E}$;*

(a_2) *If $I \in \mathcal{E}$ and $I \subseteq J$, $J \in \mathbb{L}(_RR)$, then $J \in \mathcal{E}$;*

(a_3) *If $I, J \in \mathcal{E}$, then $I \cap J \in \mathcal{E}$.*

There exists a monotone bijection between the pretorsions of R-Mod and preradical filters of $\mathbb{L}(_RR)$ defined by the mappings:

$$r \rightsquigarrow \mathcal{E}_r, \quad \mathcal{E}_r = \{I \in \mathbb{L}(_RR) \mid r(R/I) = R/I\};$$

$$\mathcal{E} \rightsquigarrow r_{\mathcal{E}}, \quad r_{\mathcal{E}}(M) = \{m \in M \mid (0 : m) \in \mathcal{E}\} \text{ ([3],[4]).}$$

We denote by $\mathbb{PF}$ the set of all preradical filters of the lattice $\mathbb{L}({}_RR)$ of left ideals of R. The sets $\mathbb{PT}$ and $\mathbb{PF}$ can be considered as complete lattices and the mappings indicated above determine an isomorphism of these lattices: $\mathbb{PT} \cong \mathbb{PF}$.

We mention that in the lattice $\mathbb{PT}$ the product of two pretorsions $r \cdot s$ coincides with their meet $r \wedge s$.

In $\mathbb{PT}(\wedge, \vee)$ we also have the operation $r \# s$ defined by the rule $[(r \# s)(M)]/s(M) = r(M/s(M))$, $M \in R$-Mod and $r \# s$ is called the *coproduct* of pretorsions r and s.

In a similar way is introduced in $\mathbb{PF}$ the notion of coproduct:
$\mathcal{E}_r \# \mathcal{E}_s = \{I \in \mathbb{L}({}_RR) \mid \exists\, H \in \mathcal{E}_r,\ I \subseteq H$ such that $(I : a) \in \mathcal{E}_s, \forall a \in H\}$.

So we have the isomorphic lattices $\mathbb{PT}(\wedge, \vee, \#)$ and $\mathbb{PF}(\bigwedge, \bigvee, \#)$ with the following properties:

$$\mathcal{E}_{\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha} = \bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}; \quad \mathcal{E}_{\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha} = \bigvee_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}.$$

Now we remind some notions and results of the monograph [1], where the pretorsions of R-Mod are investigated by the point of view of the associated preradical filters. In [1] $\mathbb{PF}$ is denoted by $R - fil$ and the operation of *multiplication* in $R - fil$ is defined by the rule:
$KK' = \{I \in \mathbb{L}({}_RR) \mid \exists\, H \in K'$, such that $I \subseteq H$ and $(I : a) \in K, \forall a \in H\}$, where $K, K' \in R - fil$.

It is easy to see that in our notations for every $r, s \in \mathbb{PT}$ we have $\mathcal{E}_s\mathcal{E}_r = \mathcal{E}_r \# \mathcal{E}_s$. All properties of the operation of multiplication easily can be translated in the language of coproduct, in particular associativity and distributivity:

$$\mathcal{E}_1 \# (\mathcal{E}_2 \# \mathcal{E}_3) = (\mathcal{E}_1 \# \mathcal{E}_2) \# \mathcal{E}_3; \quad (\bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}) \# \mathcal{E} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{E}_{r_\alpha} \# \mathcal{E}).$$

Using the product KK' of preradical filters, in [1] is defined *right residual* $K'^{-1}K$ of K by K' as the unique minimal preradical filter K'' in $R - fil$ satisfing $K'K'' \supseteq K$. By the distributivity such a filter always exists and is equal to $\bigcap\{K'' \mid K'K'' \supseteq K\}$. In the book [1] a series of properties of this operation is exposed.

Translating in our notations and making the necessary changes (multiplication versus coproduct) we obtain the following statements.

Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{PF}$. *Left coquotient with respect to meet* of $\mathcal{E}_1$ by

$\mathcal{E}_2$ is called the minimal preradical filter $\mathcal{E}$ such that $\mathcal{E} \# \mathcal{E}_2 \supseteq \mathcal{E}_1$ or $\bigwedge\{\mathcal{E} \in \mathbb{PF} \,|\, \mathcal{E} \# \mathcal{E}_2 \supseteq \mathcal{E}_1\}$. The distributivity ensures the existence of this coquotient, denoted by $\mathcal{E}_1 \,{}^{\wedge}\!/_{\#}\, \mathcal{E}_2$ ([2]).

Now we will show that this preradical filter coincides with the preradical filter of the pretorsion $r_{\mathcal{E}_1} {}^{\wedge}\!/_{\#}\, r_{\mathcal{E}_2}$.

Lemma. *If* r, $s \in \mathbb{PT}$, *then* $\mathcal{E}_{r \# s} = \mathcal{E}_r \# \mathcal{E}_s$.

Proof. Firstly we specify the expresions of pretorsions determined by indicated preradical filters, using that $r(M) = \{m \in M \mid (0 : m) \in \mathcal{E}_r\}$ for every $r \in \mathbb{PT}$ and $M \in R$-Mod.

The preradical filter $\mathcal{E}_{r \# s}$ is determined by the pretorsion $r \# s$ and $(r \# s)(M) = \{m \in M \,|\, (m + s(M)) \in r(M/s(M))\}$. But $r(M/s(M)) = \{x + s(M) \,|\, x \in M$ and $(0 : (x + s(M))) \in \mathcal{E}_r\} =$ $= \{x + s(M) \,|\, x \in M$ and $(s(M) : x) \in \mathcal{E}_r\}$, so we have $(r \# s)(M) =$ $= \{m \in M \,|\, (s(M) : m) \in \mathcal{E}_r\}$.

We denote by t the pretorsion of R-Mod defined by $\mathcal{E}_r \# \mathcal{E}_s$, so for every $M \in R$-Mod we have $t(M) = \{m \in M \,|\, (0 : m) \in \mathcal{E}_r \# \mathcal{E}_s\} =$ $= \{m \in M \,|\, \exists H \in \mathcal{E}_r,\ (0 : m) \subseteq H$ such that $((0 : m) : a) \in \mathcal{E}_s, \forall a \in H\} =$ $= \{m \in M \,|\, \exists H \in \mathcal{E}_r,\ (0 : m) \subseteq H$ such that $(0 : am) \in \mathcal{E}_s$, $\forall a \in H\}$.

Now we verify the equality of lemma.

$(\subseteq)$ It is sufficient to show that $r \# s \leq t$. For every $M \in R$-Mod if $m \in (r \# s)(M)$, then $H = (s(M) : m) \in \mathcal{E}_r$ and $(0 : m) \subseteq$ $\subseteq (s(M) : m) = H$. So if $a \in H$, then $am \in s(M)$, i.e. $(0 : am) \in \mathcal{E}_s$, which means that $m \in t(M)$. Therefore $(r \# s)(M) \subseteq t(M)$ for every $M \in R$-Mod, i.e. $r \# s \leq t$, which implies $\mathcal{E}_{r \# s} \subseteq \mathcal{E}_r \# \mathcal{E}_s$.

$(\supseteq)$ We verify that $t \leq r \# s$. Let $M \in R$-Mod and $m \in t(M)$. Then there exists $H \in \mathcal{E}_r$ such that $(0 : m) \subseteq H$ and $(0 : am) \in \mathcal{E}_s, \forall a \in H$. If $a \in H$, then $(0 : am) \in \mathcal{E}_s$, so $am \in s(M)$, i.e. $a \in (s(M) : m)$, therefore $H \subseteq (s(M) : m)$. From the definition of preradical filter (condition (a_2)) since $H \in \mathcal{E}_r$ now we have $(s(M) : m) \in \mathcal{E}_r$, which means that $m \in (r \# s)(M)$. This proves that $t(M) \subseteq (r \# s)(M)$ for every $M \in R$-Mod, therefore $t \leq r \# s$ and so $\mathcal{E}_r \# \mathcal{E}_s \subseteq \mathcal{E}_{r \# s}$. $\square$

Proposition. *For every pretorsions* r, $s \in \mathbb{PT}$ *we have:*

$$\mathcal{E}_{r \,{}^{\wedge}\!/_{\#}\, s} = \mathcal{E}_r \,{}^{\wedge}\!/_{\#}\, \mathcal{E}_s.$$

Proof. $(\supseteq)$ By definition $\mathcal{E}_r \,{}^{\wedge}\!/_{\#}\, \mathcal{E}_s = \bigwedge\{\mathcal{E} \in \mathbb{PF} \,|\, \mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r\}$, i.e.

it is the least preradical filter with the property $\mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r$. From the Lemma $\mathcal{E}_{r \mathbin{{}^{\wedge}\!/_{\#}} s} \# \mathcal{E}_s = \mathcal{E}_{(r \mathbin{{}^{\wedge}\!/_{\#}} s) \# s}$ and since $(r \mathbin{{}^{\wedge}\!/_{\#}} s) \# s \geq r$ ([2]) we have $\mathcal{E}_{(r \mathbin{{}^{\wedge}\!/_{\#}} s) \# s} \supseteq \mathcal{E}_r$, so $\mathcal{E}_{r \mathbin{{}^{\wedge}\!/_{\#}} s} \# \mathcal{E}_s \supseteq \mathcal{E}_r$. Therefore $\mathcal{E}_{r \mathbin{{}^{\wedge}\!/_{\#}} s}$ is one of preradical filter $\mathcal{E}$ and so $\mathcal{E}_{r \mathbin{{}^{\wedge}\!/_{\#}} s} \supseteq \bigwedge\{\mathcal{E} \in \mathbb{PF} \mid \mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r\}$, i.e. $\mathcal{E}_{r \mathbin{{}^{\wedge}\!/_{\#}} s} \supseteq \mathcal{E}_r \mathbin{{}^{\wedge}\!/_{\#}} \mathcal{E}_s$.

$(\subseteq)$ Let $\mathcal{E}_t$ be preradical filter defined by pretorsion t with the property $\mathcal{E}_t \# \mathcal{E}_s \supseteq \mathcal{E}_r$. From the Lemma $\mathcal{E}_{t \# s} \supseteq \mathcal{E}_r$, therefore $t \# s \geq r$. Since $r \mathbin{{}^{\wedge}\!/_{\#}} s$ is the least pretorsion h with the property $h \# s \geq r$ ([2]) it follows that $r \mathbin{{}^{\wedge}\!/_{\#}} s \leq t$ i.e. $\mathcal{E}_{r \mathbin{{}^{\wedge}\!/_{\#}} s} \subseteq \mathcal{E}_t$. So $\mathcal{E}_{r \mathbin{{}^{\wedge}\!/_{\#}} s}$ is the least between preradical filters $\mathcal{E}$ with the property $\mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r$. $\square$

As a conclusion we can affirm that all results of J.S.Golan [1] about the operation of right residual of preradical filters can be treated as a particular case of the operation of left coquotient with respect to meet, defined in [2] in general case of preradicals of $M \in R\text{-}\mathrm{Mod}$.

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On some new characterizations of pretorsions of a module category

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Abstract

Three new methods of descriptions of pretorsions in modules are shown. Pretorsions are characterized by closure operators in $\mathbb{L}({}_R R)$, by closure operators of R-Mod and by functions which distinguish the dense submodules.

Keywords: category, module, pretorsion, closure operator.

The *pretorsion* r of a module category R-Mod is a preradical with the property: $r(M) \cap N = r(N)$ for every submodule $N \subseteq M$. Pretorsions of R-Mod can be described by the classes of modules and by the filters of left ideals of R [1-3]. The aim of this communication is to show some new characterizations of pretorsions by the closure operators of special types.

Every pretorsion of R-Mod defines the class of modules $\mathcal{T}_r = \{M \in R\text{-Mod} \mid r(M) = M\}$ and the set of left ideals of R:

$$\mathcal{E}_r = \{I \in \mathbb{L}({}_R R) \mid r(R/I) = R/I\}.$$

Moreover, $\mathcal{T}_r$ and $\mathcal{E}_r$ reestablish the pretorsion r.

In the set of pretorsions $\mathbb{PT}$ of R-Mod three operations are considered: meet $(\wedge)$, join $(\vee)$ and coproduct $(\#)$ and their expressions by the classes $\mathcal{T}_r$ and the filters $\mathcal{E}_r$ are indicated.

Proposition 1. *For the pretorsions of R-Mod the following relations are true:*

1) $\mathcal{T}_{\bigwedge\limits_{\alpha \in \mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha \in \mathfrak{A}} \mathcal{T}_{r_\alpha}, \qquad \mathcal{T}_{\bigvee\limits_{\alpha \in \mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha \in \mathfrak{A}} \mathcal{T}_{r_\alpha}, \qquad \mathcal{T}_{r \# s} = \mathcal{T}_r \# \mathcal{T}_s;$

2) $\mathcal{E}_{\bigwedge\limits_{\alpha \in \mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha \in \mathfrak{A}} \mathcal{E}_{r_\alpha}, \qquad \mathcal{E}_{\bigvee\limits_{\alpha \in \mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha \in \mathfrak{A}} \mathcal{E}_{r_\alpha}, \qquad \mathcal{E}_{r \# s} = \mathcal{E}_r \# \mathcal{E}_s.$

In the lattice of left ideals $\mathbb{L}({}_RR)$ of R the *modular preclosure operators* are defined by the conditions of extension, monotony, modularity and linearity.

Proposition 2. *There exists a monotone bijection between the pretorsions of R-Mod and the modular preclosure operators of $\mathbb{L}({}_RR)$.*

The other form of characterization of pretorsions is obtained by the closure operators of the category R-Mod ([4]).

Proposition 3. *There exists a monotone bijection between the pretorsions of R-Mod and the maximal and hereditary closure operators of R-Mod.*

The description of pretorsions by the dense submodules (with respect to some closure operator of R-Mod) also is obtained.

Proposition 4. *There exists a monotone bijection between the pretorsions of R-Mod and the abstract functions of type $\mathfrak{F}_1$ (see [4]), which are maximal and hereditary.*

Some approximations of pretorsions by jansian pretorsions and by torsions are constructed.

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On parastrophes of quasigroup identities and corresponding varieties and trusses

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Abstract

V.D. Belousov [1] classified minimal nontrivial quasigroup identities up to parastrophically primary equivalency [2] and obtained seven identities. Here, the kinds of trusses of the corresponding trusses of varieties are defined. Identities for each of the varieties are found. Identities defining semi-symmetric trusses are described. The existence of these classes was considered a problem [5].

Keywords: quasigroup, loop, parastrophe, identity, variety, truss.

1 Introduction

Recall [3] that the σ-parastrophe of a class $\mathfrak{A}$ of quasigroups is called the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$ and is denoted by ${}^{\sigma}\mathfrak{A}$. The set of all parastrophes of a class $\mathfrak{A}$ is called a truss of $\mathfrak{A}$ and is denoted by $\mathrm{tr}(\mathfrak{A})$. Since $|\mathrm{tr}(\mathfrak{A})|$ divides $|S_3|$ (S_3 is the group of all permutations of $\{1,2,3\}$) then there are four possibilities: $|\mathrm{tr}(\mathfrak{A})| \in \{1,2,3,6\}$.

Let $\mathrm{Ps}(\mathfrak{A})$ denote the set of all σ such that ${}^{\sigma}\mathfrak{A} = \mathfrak{A}$. $\mathrm{Ps}(\mathfrak{A})$ is a subgroup of S_3 and is called the symmetry group of $\mathfrak{A}$. A truss $\mathrm{tr}(\mathfrak{A})$ is said to be:

– *totally symmetric*, if it has one element: $\mathrm{tr}(\mathfrak{A}) = \{\mathfrak{A}\}$, i.e., all parastrophes of $\mathfrak{A}$ coincide, therefore $\mathrm{Ps}(\mathfrak{A}) = S_3$;

– *semi-symmetric*, if it has two elements: $\mathrm{tr}(\mathfrak{A}) = \{\mathfrak{A}, {}^{(12)}\mathfrak{A}\}$, i.e.,

$\mathrm{Ps}(\mathfrak{A}) = \mathrm{Ps}({}^{(12)}\mathfrak{A}) = A_3$;
– *middle symmetric*, if it has three elements $\mathrm{tr}(\mathfrak{A}) = \{\mathfrak{A}, {}^{(13)}\mathfrak{A}, {}^{(23)}\mathfrak{A}\}$, i.e., their symmetry groups are $\{\iota, (12)\}$, $\{\iota, (13)\}$, $\{\iota, (23)\}$;
– *asymmetric*, if it has six elements, i.e., all parastrophes are pairwise different, in other words, their symmetry group is identical, that is $\{\iota\}$.

Let $\mathfrak{id}$ be an identity. An identity obtained from $\mathfrak{id}$ by replacing the main operation with its σ^{-1}-parastrophe is called σ-parastrophe of $\mathfrak{id}$ and is denoted by ${}^{\sigma}\mathfrak{id}$. A σ-parastrophic identity defines a σ-parastrophic variety.

2 Main results

V.D. Belousov [1] described minimal nontrivial quasigroup identities up to parastrophiccally primary equivalency [2] and obtained the following seven identities: **Belousov I:** $x(x{\cdot}xy) = y$; **Belousov II:** $y(x{\cdot}xy) = x$;, **Stein I:** $x \cdot xy = yx$; **Stein II:** $xy \cdot x = y \cdot xy$; **Stein III:** $yx \cdot xy = x$; **Shröder I:** $xy \cdot y = x \cdot xy$; **Shröder II:** $xy \cdot yx = y$.

The following question is natural: what kind of trusses do they define?

Table 1. Middle symmetric varieties with $\mathrm{Ps} = \{\iota, (23)\}$

Parastrophes	$\mathfrak{A} = {}^{r}\mathfrak{A}$	${}^{s}\mathfrak{A} = {}^{sr}\mathfrak{A}$	${}^{\ell}\mathfrak{A} = {}^{s\ell}\mathfrak{A}$
Belousov I	$x(x{\cdot}xy) = y$	$(yx \cdot x)x = y$	$x(yx/y) = yx$ $(x\backslash yx)y = yx$

Table 2. Middle symmetric varieties with $\mathrm{Ps} = \{\iota, (12)\}$

Parastrophes	$\mathfrak{A} = {}^{s}\mathfrak{A}$	${}^{\ell}\mathfrak{A} = {}^{sr}\mathfrak{A}$	${}^{r}\mathfrak{A} = {}^{s\ell}\mathfrak{A}$
Stein II	$xy{\cdot}x = y{\cdot}xy$	$y(x \cdot yx) = x$	$(xy \cdot x)y = x$
Stein III	$yx \cdot xy = x$	$(xy{\cdot}x)xy = y$, $y(xy \cdot x) = x$	$xy(y \cdot xy) = x$ $(x \cdot yx)y = x$

Table 3. Asymmetric varieties

Parastrophes	of Belousov's law II
$\mathfrak{A}$	$y(x \cdot xy) = x, \quad xy(xy \cdot x) = y$
${}^{s}\mathfrak{A}$	$(yx \cdot x)y = x, \quad (x \cdot yx)yx = y$
${}^{\ell}\mathfrak{A}$	$x(yx \cdot y) = yx, \quad y(xy \cdot x) = xy$
${}^{r}\mathfrak{A}$	$x(xy \cdot x) = y, \quad x(x \cdot yx) = y, \quad (x \cdot xy)x = y$
${}^{s\ell}\mathfrak{A}$	$(xy \cdot x)x = y, \quad x(yx \cdot x) = y, \quad (x \cdot yx)x = y$
${}^{sr}\mathfrak{A}$	$(x \cdot yx)y = yx, \quad (y \cdot xy)x = xy$
Parastrophes	of Stein's law I
$\mathfrak{A}$	$x \cdot xy = yx$
${}^{s}\mathfrak{A}$	$yx \cdot x = xy$
${}^{\ell}\mathfrak{A}$	$x(y \cdot yx) = yx$
${}^{r}\mathfrak{A}$	$(x \cdot xy)y = x, \quad xy(xy \cdot y) = x$
${}^{s\ell}\mathfrak{A}$	$(x \cdot xy)xy = y, \quad y(yx \cdot x) = x$
${}^{sr}\mathfrak{A}$	$(xy \cdot y)x = xy$

Theorem 1. *The identities*
– Shröder I and Shröder II define totally symmetric trusses;
– Stein II, Stein III and Belousov I define middle symmetric trusses;
– Stein I and Belousov II define asymmetric trusses.

Corollary 1. *The identities of the Belousov's law II and* $xy\cdot(xy\cdot x) = y$ *are equivalent and define asymmetric variety of quasigroups.*

Table 4. Semi-symmetric varieties

Parastrophes of the varieties	$\mathfrak{A} = {}^{sr}\mathfrak{A} = {}^{s\ell}\mathfrak{A}$	${}^{s}\mathfrak{A} = {}^{r}\mathfrak{A} = {}^{\ell}\mathfrak{A}$
of quasigroups	$x^2 \cdot x = x$	$x \cdot x^2 = x$
of group isotopes	$x(yz \cdot zx) = y,$ $(xy \cdot z)zx = y,$ $x(y\cdot zx)\cdot z = y,$	$(xy \cdot yz)x = z,$ $xy(y \cdot zx) = z,$ $x\cdot(yx\cdot z)y = z.$

In the Tables 1-3, the identities in one definite sell are equivalent;

the identities in different sells are parastrophic. In Table 4, the answer to F. Sokhatsky's problem [4] is given. These varieties are not subvarieties of the variety of totally symmetric quasigroups.

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Infinite sharply 2-transitive permutation groups

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Abstract

Sharply 2-transitive permutation groups are studied in this work. The notions of transversal in a group [3] and Sabinin's semidirect product [5] are used. All elements of order 2 from a sharply 2-transitive permutation group G form a loop transversal T in G to $H_0 = St_0(G)$. For an arbitrary element $t_i \in T$ ($t_i \neq id$) its centralizer $C_i = C_G(t_i)$ is a group transversal in G to $H_0 = St_0(G)$. The group G may be represented as a semidirect product: $G = T \leftthreetimes H_0 = C_i \leftthreetimes H_0$. A construction of the group G as an external semidirect product of two suitable groups is described. It gives us a potential example of an infinite sharply 2-transitive permutation group G with a non-abelian normal subgroup T, which consists of fixed-point-free permutations and the identity permutation.

Keywords: permutation group, loop, transversal, semidirect product, centralizer.

1 Introduction

Finite sharply 2-transitive permutation groups were described by Zassenhaus [6]. He proved that a sharply 2-transitive permutation group G on a finite set of symbols E is a group G^* of linear transformations of some near-field $< E, +, \cdot >$:

$$G^* = \{\alpha_{a,b} \mid \alpha_{a,b}(t) = a \cdot t + b, \;\; a \neq 0, \;\; a, b, t \in E\}.$$

In the case when the set E is infinite, the problem of classification of sharply 2-transitive permutation groups on E is open.

2 Preliminary lemmas and a partition on cases

Let G be a sharply 2-transitive permutation group on some set E.

Lemma 1. [1] *All elements of order* 2 *from* G *are in one and the same class of conjugate elements.*

Since G is a sharply 2-transitive permutation group, then only the identity permutation **id** fixes more than one symbol from E. So we obtain the following two cases:

1) Every element of order 2 from G is a fixed-point-free permutation on E.

2) Every element of order 2 from G has exactly one fixed point from E.

Lemma 2. [1] *Let* α *and* β *be distinct elements of order* 2 *from* G*. Then the permutation* $\gamma = \alpha\beta$ *is a fixed-point-free permutation on* E*.*

Let 0 and 1 be some distinguished distinct elements from E. Denote $H_0 = St_0(G)$.

3 A loop transversal in the group G and its properties

Theorem 1. [2] *In the case 1) there exists a left loop transversal* T *in* G *to* H_0*, which consists of* ***id*** *and elements of order* 2*.*

Theorem 2. [2] *In the case 2) there exists a left non-reduced general type quasigroup transversal* T *in* G *to* H_0*, which consists of elements of order* 2*.*

Theorem 3. [2] *The transversal* T *is a normal subset in the group* G*.*

4 Centralizers of the elements of a loop transversal T and its properties

For an arbitrary (fixed) element $t_i \in T$ let us consider its centralizer $C_i = C_G(t_i)$:

$$C_i = C_G(t_i) = \{g \in G \,|\, gt_i = t_i g\}.$$

Theorem 4. [4] *The following statements are true:*

1. *For any $i \in E - \{1\}$ the set C_i is a left transversal in G to H_0,*
2. *For any $i \in E - \{1\}$ the left transversal C_i is a group transversal in G to H_0,*
3. *For any $i, j \in E - \{1\}$ transversal operations $< C_i, \cdot >$ and $< C_j, \cdot >$ are isomorphic.*

Theorem 5. [4] *For any $i, j \in E - \{1\}$ sets C_i and C_j are conjugated in G by elements from the group H_0.*

5 The group G as a semidirect product of a transversal and a subgroup H_0

5.1 Representation of the group G

Theorem 6. *The group G is a semidirect product of a loop transversal T and H_0.*

Theorem 7. *For any $i \in E - \{1\}$ the group G is a semidirect product of subgroups C_i and H_0.*

5.2 Construction of the group G as an external semidirect product of two suitable groups

Theorem 8. *Let $T = \langle E, \bullet, 1\rangle$ be a group and $H = Inn(T)$ be the group of all inner automorphisms of T. Then there exists a transitive permutation group G, which is a semidirect product $G = T \leftthreetimes H$.*

Theorem 9. *If in the previous proposition T is an infinite non-abelian simple group, then:*

1. *G is an infinite sharply 2-transitive permutation group;*

2. *Fixed-point-free permutations of the group G with the identity permutation* **id** *form a normal subgroup T_0, which is isomorphic to the (non-abelian) group T.*

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Morita equivalence of semigroups revisited

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Abstract

Firm semigroups and firm acts are defined as non-additive analogues of firm rings and firm modules. Using the categories of firm acts Morita theory is developed for firm semigroups. The main result states that the categories of firm right acts over two firm semigroups are equivalent if and only if these semigroups are strongly Morita equivalent, which means that they are contained in a unitary Morita context with bijective mappings.

Other categories of acts which have been used earlier to develop Morita equivalence are also investigated. Over firm semigroups all the considered categories turn out to be equivalent to the category of firm acts.

Keywords: Firm semigroup, firm act, Morita equivalence, strong Morita equivalence, adjoint functors.

1 Introduction

A theory of Morita equivalence was carried over from unital rings to monoids independently by Banaschewski and Knauer in the early seventies but their results have not really been taken up. In the eighties, Morita equivalence was extended to much wider classes of rings, no longer requiring the existence of an identity element. To construct a useful theory of Morita equivalence in the non-unital case, one had to restrict both the class of rings and the class of modules to be considered. Talwar [9] found a viable approach to Morita equivalence for semigroups without identity but with certain idempotents called local units, showing also the relevance of Morita equivalence in the structure theory of semigroups. He also extended some results to factorisable semigroups (those in which every element decomposes as a product). Subsequently, Chen and Shum as well as Neklyudova contributed to the theory.

A decisive step was made by Lawson [5] in 2011. He considered the class of semigroups with local units, defined in the same way as by Talwar. However, instead of Talwar's fixed acts he considered closed acts – these are easier to get around with than fixed acts, and Lawson proved that these two kinds of acts coincide over semigroups with local units. The main result in Lawson's work is the fact that, for semigroups with local units, every Morita equivalence is strong in the sense that it comes from a unitary Morita context with surjective mappings. Lawson [5] as well as Laan and Márki [4] give various structural characterisations of Morita equivalence for semigroups with local units. By all this one can say that we have a satisfactory theory of Morita equivalence for semigroups with local units. Attempts to extend the theory beyond semigroups with local units have not brought decisive results so far concerning the problem whether every Morita equivalence is strong.

Here we consider the same class of acts as was done in [5] and [4]. There they were called 'closed acts' – here we call them 'firm acts'. Namely, these acts are exactly the non-additive analogues of modules called 'firm modules' by Quillen [7], used later also in many papers by Marín. We call a semigroup 'firm' if it is a firm act over itself. The main result of our paper is that two firm semigroups are strongly Morita equivalent if and only if the categories of firm right acts over these semigroups are equivalent. We also consider other categories of acts used for building Morita theory by other authors, as well as categories of acts which correspond to categories of modules used by García and Marín [2], with the aim of clarifying the relations between these categories. Our main tool is the usage of adjunctions between various categories of acts. In our eyes, the results in the present paper are convincing enough to claim that firm semigroups and firm acts are the natural environment to study Morita equivalence of acts.

2 Preliminary notions

A semigroup S is called **factorisable** if every element of S is a product of two elements. We say that an element s of a semigroup has a **weak right local unit** u (**weak left local unit** v) if $su = s$ ($vs = s$). A semigroup S has **weak local units** if each of its elements has both a weak right and a weak left local unit, **local units** if the elements u, v above can always be chosen to be idempotent, and **common weak local units** if for every $s, t \in S$ there exist $u, v \in S$ such that $s = su = vs$, $t = tu = vt$.

Let S and T be semigroups. We use the notation Act_S (${}_S\mathsf{Act}$, ${}_S\mathsf{Act}_T$) for the category of all right S-acts (left S-acts, (S,T)-biacts) where morphisms are right S-act homomorphisms (left S-act homomorphisms, (S,T)-biact homomorphisms). A right S-act A_S is called **unitary** if $AS = A$. We denote the category of all unitary right S-acts by UAct_S.

We say that a right S-act A_S is **firm** if the mapping

$$\mu_A : A \otimes S \to A, \quad a \otimes s \mapsto as$$

is bijective. A semigroup S is called **firm** if it is firm as a right (or, equivalently, left) S-act. The category of all firm right S-acts is denoted by FAct_S.

Obviously, A_S is unitary if and only if the mapping μ_A is surjective. Hence, for any semigroup S, FAct_S is a subcategory of UAct_S. Also, a semigroup S is factorisable if and only if μ_S is surjective.

A right S-act A_S is called **nonsingular** if $a = a'$ ($a, a' \in A$) whenever $as = a's$ for all $s \in S$. Denote the category of unitary nonsingular right S-acts by NAct_S. This category is used for developing Morita theory by Chen and Shum [1].

What we know is that $\mathsf{FAct}_S = \mathsf{NAct}_S$ if S is a semigroup with common weak local units. However, even for semigroups with local units, these categories need not coincide.

3 Categories of acts over firm semigroups

We show that the different settings in which Morita equivalence of semigroups is considered in earlier papers amount to the same equivalence for firm semigroups.

Proposition 1. *Let S and T be semigroups and let ${}_SP_T \in {}_S\mathsf{Act}_T$. Then the functor $- \otimes P \colon \mathsf{Act}_S \to \mathsf{Act}_T$ is left adjoint to the functor $\mathsf{Act}_T(P, -) \colon \mathsf{Act}_T \to \mathsf{Act}_S$.*

Proposition 2. *Let S be a firm semigroup. Then $A \otimes S$ is a firm right S-act for any right S-act A_S and $- \otimes S$ is a functor $\mathsf{Act}_S \to \mathsf{FAct}_S$.*

It is easy to check that $\mu : -\otimes S \to 1_{\mathsf{Act}_S}$ is a natural transformation.

In the case of modules, a category denoted by CMod-R (see, for example, [2]) has been used in investigations of Morita equivalence mainly in the works of Marín. Its act counterpart would be the full subcategory of Act_S given by the acts A_S for which the natural map $\lambda_A \colon A \to \mathsf{Act}_S(S, A)$ which corresponds to $\mu_A \colon A \otimes S \to A$ under the adjunction $- \otimes S \dashv \mathsf{Act}_S(S, -)$ is invertible. We will denote this category by CAct_S.

Next, if S is a factorisable semigroup and A_S is a right S-act, then $AS = \{as \mid a \in A, s \in S\}$ is the largest unitary subact of A_S. This construction is functorial.

There is a remarkably strong result in [6] which says that, for an idempotent ring (the counterpart of a factorisable semigroup in ring theory), the category of firm modules is equivalent to the category Mod-R whose objects are modules M_R such that $MR = M$ and, for all $m \in M$, if $mR = 0$ then $m = 0$ (the counterpart of NAct_S), and also to the category CMod-R. In the semigroup case we have:

Theorem 3. *For a firm semigroup S, the categories FAct_S, NAct_S and CAct_S are equivalent.*

We can combine the information about firm acts to obtain the following result.

Theorem 4. *Let S be a firm semigroup and A_S a unitary right S-act. Then the following assertions are equivalent.*

1. *A_S is firm.*
2. *There exists an isomorphism $A \otimes S \to A$ of right S-acts.*
3. *ε_A is invertible.*
4. *ε'_A is invertible.*

Here ε_A and ε'_A are the counits of certain adjunctions.

4 Morita contexts and the main result.

In [5] Lawson showed that, for a semigroup S with local units, the category ${}_S\mathsf{FAct}$ coincides with the category consisting of left S-acts ${}_SA$ for which the canonical mapping $S \otimes S\mathsf{Act}(S, A) \to A$ is bijective. Acts fixed by ε over semigroups with local units were introduced by Talwar in [9]. In a subsequent paper [10], he used acts fixed by the map ε' above to develop Morita theory for factorisable semigroups. So Theorem 4 yields immediately the following generalisation of Lawson's result to firm semigroups.

Corollary 5. *Over a firm semigroup, firm acts are the same as fixed acts in the sense of Talwar [9] and [10].*

Next we have an analogue of the Eilenberg-Watts theorem, stating that equivalence functors between categories of firm acts over firm semigroups are naturally isomorphic to tensor multiplication functors.

Theorem 6. *Let S and T be firm semigroups and let $F\colon \mathsf{FAct}_S \to \mathsf{FAct}_T$ and $G\colon \mathsf{FAct}_S \to \mathsf{FAct}_T$ be mutually inverse equivalence functors.*

Then

$$F \cong - \otimes F(S),$$
$$G \cong - \otimes G(T).$$

Moreover, the left acts ${}_SF(S)$ *and* ${}_TG(T)$ *are firm.*

Finally, we have that right Morita equivalence and strong Morita equivalence coincide on the class of firm semigroups.

A **Morita context** is a six-tuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, where S and T are semigroups, ${}_SP_T \in {}_S\mathsf{Act}_T$ and ${}_TQ_S \in {}_T\mathsf{Act}_S$ are biacts, and

$$\theta : {}_S(P \otimes Q)_S \to {}_SS_S, \quad \phi : {}_T(Q \otimes P)_T \to {}_TT_T$$

are biact morphisms such that, for every $p, p' \in P$ and $q, q' \in Q$,

$$\theta(p \otimes q)p' = p\phi(q \otimes p'), \quad q\theta(p \otimes q') = \phi(q \otimes p)q'.$$

We say that a Morita context $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$ is

- **unitary**, if ${}_SP_T$ and ${}_TQ_S$ are unitary biacts,
- **surjective**, if θ and ϕ are surjective,
- **bijective**, if θ and ϕ are bijective.

Semigroups S and T are called **strongly Morita equivalent** if they are contained in a unitary surjective Morita context.

While it is clear what is meant by strong Morita equivalence, it is not so obvious what right Morita equivalence should mean. In different articles, various categories have been used to define Morita equivalence. In the present text we have shown that, at least for firm semigroups, it does not make any difference which of these categories one uses. For us, the category of firm acts seems to be the most natural choice.

We say that semigroups S and T are **right Morita equivalent** if the categories FAct_S and FAct_T are equivalent.

One of the central questions in Morita theory is: when right Morita equivalence and strong Morita equivalence coincide. By [5], they coincide on the class of semigroups with local units. Our main result is a far-reaching generalisation of Lawson's theorem. It is the non-additive counterpart of a theorem announced by Quillen [8] in 1996 and rediscovered by García and Marín in [2], stating that any Morita equivalence between firm rings is given by a unique Morita context.

Theorem 7. *Let S and T be firm semigroups. The following assertions are equivalent.*

1. *The categories FAct_S and FAct_T are equivalent.*
2. *The categories ${}_S\mathsf{FAct}$ and ${}_T\mathsf{FAct}$ are equivalent.*
3. *There exists a unitary bijective Morita context containing S and T.*
4. *There exists a unitary surjective Morita context containing S and T.*
5. *There exists a surjective Morita context containing S and T.*

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On invariance of recursive differentiability under the isotopy of left Bol loops

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Abstract

We prove that the recursive derivatives of order 1 of isotopic left Bol loops are isotopic and that every loop, isotopic to a recursively 1-differentiable left Bol loop, is recursively 1-differentiable. The recursive differentiability of di-associative loops is also considered.

Keywords: Recursively differentiable quasigroup, recursive derivative, isostrophe, core, LIP-loop, left Bol loop

1 Introduction

Recursively s-differentiable quasigroups ($s \geq 1$) have been defined in [1], were they appear as check functions of complete recursive codes. Let $(Q, \cdot)$ be a quasigroup and let i be a natural number. The operation $(\overset{i}{\cdot})$, defined recursively on Q as follows:

$$x \overset{0}{\cdot} y = x \cdot y,\ x \overset{1}{\cdot} y = y \cdot (x \cdot y),\ x \overset{i}{\cdot} y = (x \overset{i-2}{\cdot} y) \cdot (x \overset{i-1}{\cdot} y),$$

for $\forall x, y \in Q$, is called the recursive derivative of order i of $(Q, \cdot)$. A quasigroup $(Q, \cdot)$ is called recursively s-differentiable if its recursive derivatives $(Q, \overset{i}{\cdot})$ are quasigroups, for all $i = 0, 1, ..., s$. If $(Q, \cdot)$ is a loop then the grupoid $(Q, +)$, where $x + y = x \cdot (y \setminus x), \forall x, y \in Q$, is called the core of $(Q, \cdot)$. The notion of core of a loop was introduced by R. Bruck [2] for Moufung loops and studied by V. Belousov in left Bol loops [3]. It is shown in [4] that in LIP-loops the core is isostrophic to the recursive derivative of order 1.

We study the invariance of recursive differentiability under the isotopy of LIP-loops in the present work. It is proved that the recursive derivatives of order 1 of isotopic left Bol loops are isotopic and that every loop, isotopic to a recursively 1-differentiable left Bol loop, is recursively 1-differentiable. Also we show that the recursive derivative of order 1 of a recursively 1-differentiable di-associative loop is an RIP-quasigroup.

2 Recursive derivatives and cores

Let $(Q,\cdot)$ be an LIP-loop. We'll denote below by $I^{(\cdot)}$ the inversion in $(Q,\cdot): I^{(\cdot)}(x)=x^{-1}$, $\forall x\in Q$.

Theorem 1. *If the cores of two LIP-loops are isotopic then their recursive derivatives of order one are isotopic.*

Proof. Let $(Q,\oplus)$ and $(Q,+)$ be the cores and let $(Q,\overset{1}{\circ})$ and $(Q,\overset{1}{\cdot})$ be the recursive derivatives of order 1 of two loops $(Q,\circ)$ and $(Q,\cdot)$, respectively. If $(Q,\oplus)$ and $(Q,+)$ are isotopic then $\exists\alpha,\beta,\gamma\in S_Q$: $\gamma(x\oplus y)=\alpha(x)+\beta(y)\Leftrightarrow\gamma(x\circ[I^{(\circ)}(y)\circ x])=\alpha(x)\cdot[I^{(\cdot)}\beta(y)\cdot\alpha(x)]$, $\forall x,y\in Q$. Replacing $y\mapsto I^{(\circ)}(y)$, we obtain $\gamma[x\circ(y\circ x)]=\alpha(x)\cdot[I^{(\cdot)}\beta I^{(\circ)}(y)\cdot\alpha(x)]\Leftrightarrow\gamma(y\overset{1}{\cdot}x)=I^{(\cdot)}\beta I^{(\circ)}(y)\overset{1}{\cdot}\alpha(x)$, for $\forall x,y\in Q$, hence $(Q,\overset{1}{\circ})$ and $(Q,\overset{1}{\cdot})$ are isotopic.

V. Belousov [3] proved that the cores of isotopic left Bol loops are isomorphic. This fact and the previous theorem imply the following corollaries.

Corollary 1. *The recursive derivatives of order 1 of isotopic left Bol loops are isotopic.*

Corollary 2. *Every loop isotopic to a recursively* 1*-differentiable left Bol loop is recursively* 1*-differentiable.*

Proposition 1. *If the cores of two LIP-loops $(Q,\circ)$ and $(Q,\cdot)$ are isomorphic and φ is an isomorphism between them, then the recursive derivatives $(Q,\overset{1}{\circ})$ and $(Q,\overset{1}{\cdot})$ are isomorphic if and only if $I^{(\cdot)}\varphi I^{(\circ)}=\varphi$.*

Proof. Let $(Q,\oplus)$ and $(Q,+)$ be the cores and $(Q,\overset{1}{\circ})$ and $(Q,\overset{1}{\cdot})$

be the recursive derivatives of order 1 of two LIP-loops $(Q,\circ)$ and $(Q,\cdot)$, respectively. If φ is an isomorphism between the cores, then $\varphi(x\oplus y)=\varphi(x)+\varphi(y) \Leftrightarrow \varphi(x\circ(I^{(\circ)}(y)\circ x))=\varphi(x)\cdot(I^{(\cdot)}\varphi(y)\cdot\varphi(x)) \Leftrightarrow \varphi(I^{(\circ)}(y)\overset{1}{\circ} x) = I^{(\cdot)}\varphi(y)\overset{1}{\cdot}\varphi(x)$. Replacing $I^{(\circ)}(y)\mapsto y$, we obtain $\varphi(y\overset{1}{\circ} x)=I^{(\cdot)}\varphi I^{(\circ)}(y)\overset{1}{\cdot}\varphi(x)$ for $\forall x,y\in Q$. Hence φ is an isomorfism from $(Q,\overset{1}{\circ})$ to $(Q,\overset{1}{\cdot})$ if and only if $I^{(\cdot)}\varphi I^{(\circ)}=\varphi$.

Corollary 3. *If two left Bol loops $(Q,\circ)$ and $(Q,\cdot)$ are isotopic and φ is an isomorphism between their cores then $(Q,\overset{1}{\circ})$ and $(Q,\overset{1}{\cdot})$ are isomorphic if and only if $I^{(\cdot)}\varphi I^{(\circ)}=\varphi$.*

Theorem 2. *Let $(Q,\cdot)$ be a di-associative loop. Then $(Q,\cdot)$ is recursively differentiable if and only if the mapping $x\mapsto x^2$ is a bijection. Moreover, if $(Q,\cdot)$ is recursively 1-differentiable then its recursive derivative of order 1 is an RIP-quasigroup.*

Proof. Let $(Q,\cdot)$ be a di-associative loop. Its recursive derivative of order 1, $(Q,\overset{1}{\cdot})$ is a quasigroup if the equations $a\overset{1}{\cdot}x=b$ and $y\overset{1}{\cdot}a=b$ have unique solutions in Q for all $a,b\in Q$. But $y\overset{1}{\cdot}a=b\Leftrightarrow a\cdot ya=b$ has a unique solution in Q, for all $a,b\in Q$. Hence $(Q,\cdot)$ is recursively 1-differentiable if and only if

$$a\overset{1}{\cdot}x=b\Leftrightarrow x\cdot ax=b\Leftrightarrow ax\cdot ax=ab\Leftrightarrow (ax)^2=ab$$

has a unique solution in Q, for all $a,b\in Q$, i.e. if and only if the mapping $x\mapsto x^2$ is a bijection.

If $(Q,\cdot)$ is recursively 1-differentiable, then $(Q,\overset{1}{\cdot})$ is a quasigroup, and $(y\overset{1}{\cdot}x)\overset{1}{\cdot}x^{-1}=x^{-1}\cdot(x\cdot yx)\cdot x^{-1}=y$, as $(Q,\cdot)$ is a di-associative loop. Hence $(Q,\overset{1}{\cdot})$ is an RIP-quasigroup.

Proposition 2. [5] *Let $(Q,\cdot)$ be a quasigroup and let $(Q,\circ)$ be a quasigroup with a right unit e. If $x\circ y=\varphi(y)\cdot\psi(x)$, for $\forall x,y\in Q$, and $\psi\in Aut(Q,\circ)$, then $RM(Q,\circ)\triangle LM(Q,\cdot)$.*

Corollary 4. *If $(Q,\cdot)$ is a recursively 1-differetiable left Bol loop, $(Q,+)$ is its core and $(Q,\overset{1}{\cdot})$ is its recursive derivative of order 1, then $RM(Q,\overset{1}{\cdot})\triangle LM(Q,+)$.*

Proof. Let $(Q,\cdot)$ be a recursively 1-differetiable left Bol loop, then its core $(Q,+)$ and its recursive derivative of order 1 $(Q,\overset{1}{\cdot})$ are quasigroups. It is shown in [4] that: the unit of $(Q,\cdot)$ is the right unit of $(Q,\overset{1}{\cdot})$, $x+y=I^{(\cdot)}(y)\overset{1}{\cdot}x$ and that, if $(Q,\cdot)$ is a left Bol loop, then $I^{(\cdot)}(x\overset{1}{\cdot}y)=I^{(\cdot)}(x)\overset{1}{\cdot}I^{(\cdot)}(y)$. Hence the quasigroups $(Q,+)$ and $(Q,\overset{1}{\cdot})$, where $x+y=I^{(\cdot)}(y)\overset{1}{\cdot}x$ and $I^{(\cdot)}\in Aut(Q,\overset{1}{\cdot})$, fulfill the conditions of the previous proposition, hence $RM(Q,\overset{1}{\cdot})\triangle LM(Q,+)$.

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Goal-Driven Machine Proof Search in Intuitionistic First-Order Logic

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Abstract

A sequent calculus is proposed for machine-oriented proof search in intuitionistic first-order logic without equality. One of its distinctive features is that for optimizing quantifier handling, the author's original notions of admissibility and compatibility of substitutions of terms for variables are used. Another is that the selection of some proportional rules for its application in an inference tree is "driven" by a formula from the succedent of a sequent under consideration. The proposed calculus is sound and complete and can serve as a basis for constructing provers for proof search in intuitionistic first-order logic.

Keywords: intuitionistic first-order logic, admissibility, compatibility, validity, completeness, machine proof search.

1 Introduction

In a number of the author's papers (see, for example, [1]), an approach is developed for optimizing quantifier handling in sequential calculi. It bases on the original notions of the admissibility of a substitution of terms for variables and its compatibility with a (constructed) sequent inference tree and leads to improving the efficiency of machine proof search in nonclassical first-order logics, in particular, in intuitionistic logic. However, in that papers there are no recommendations for optimizing orders of propositional rule applications. The purpose of this work is to show that for intuitionistic logic without equality, one can construct a sequential calculus (that uses the author's admissibility and compatibility instead of the Gentzen [2] and Kanger [3] ones) giving also a certain ("goal-driven") optimization of orders of certain propositional rule applications.

2 Goal-driven sequent calculus

We refer to [1] and the papers cited therein for all precise notions and definitions that are used in our sequent treatment of intuitionistic logic without equality and are not formally introduced here. This treatment as a sequent calculus denoted by $\mathbf{LJ}^*$ and combining some of the ideas of G. Gentzen and S. Kanger is given in Figure 1. The calculus $\mathbf{LJ}^*$ is intended for the establishing of the deducibility of a sequent $\rightarrow F$ in the Gentzen intuitionistic calculus $\mathbf{LJ}$ [2] (F is a usual closed formula).

The notion of a *positive* and *negative occurrence* of a formula and quantifier in a formula and sequent is used in the usual sense. If a fixed occurrence of a formula G in a formula F from a sequent has a positive (negative) occurrence, then we write $F\langle^{+}G\rangle$ ($F\langle^{-}G\rangle$). Following [3], the variable of a positive quantifier is called a *parameter* and the variable of a negative quantifier is called a *dummy*.

Let A and A' be atomic formulas of the forms $P(t_1, \ldots, t_n)$ and $P(t'_1, \ldots, t'_n)$ respectively, where P is a predicate symbol and $t_1, \ldots, t_n, t'_1, \ldots, t'_n$ are terms. If there exists a most general simultaneous unifier (mgsu) σ of the sets $\{t_1, t'_1\}, \ldots, \{t_n, t'_n\}$, then this fact is denoted by $A \approx A'$ w.r.t. the mgsu σ.

For each usual propositional (quantifier) connective $\odot$ (Qx, where Q is $\forall$ or $\exists$ and x is a variable) we introduce the countable set of its "copies" ${}^1_j\odot, {}^2_j\odot, \ldots, {}^i_j\odot, \ldots$ (${}^1_jQ^1_jx, {}^2_jQ^2_jx, \ldots, {}^i_jQ^i_jx, \ldots$) with the same semantic interpretation that $\odot$ (Qx) has, where $j = 1, 2, \ldots$.

The establishing of the deducibility of a sequent $\rightarrow F$ in $\mathbf{LJ}$ is replaced by the establishing of the deducibility of the so-called *starting sequent* $\rightarrow {}^1[{}_\nu F]$, where ${}^1[{}_\nu F]$ is constructed according to [1]: if $\odot$ (Qx) occurs in F and occupies in F the nth position when looking through F from left to right, then ${}^1[{}_\nu F]$ contains ${}^1_n\odot$ (${}^1_nQ^1_nx$) instead of $\odot$ (Qx); at that, all the other occurrences of x in ${}^1[{}_\nu F]$ become ${}^1_nx$.

$$\mathit{Axioms:}$$

$$\frac{}{\Gamma, A \to A}$$

$$\mathit{Goal\ splitting\ rules:}$$

$$\frac{\Gamma, F \to}{\Gamma \to {}^{i}_{n}\neg F}\ (\to \neg) \qquad \frac{\Gamma, F \to G}{\Gamma \to F \,{}^{i}_{n}\!\supset G}\ (\to \supset)$$

$$\frac{\Gamma \to F \quad {}^{l+1}[\Gamma] \to {}^{l+1}[G]}{\Gamma \to F \,{}^{i}_{n}\!\wedge G}\ (\to \wedge)$$

$$\frac{\Gamma \to F}{\Gamma \to F \,{}^{i}_{n}\!\vee G}\ (\to \vee_1) \qquad \frac{\Gamma \to G}{\Gamma \to F \,{}^{i}_{n}\!\vee G}\ (\to \vee_2)$$

$$\mathit{Premise\ splitting\ rules:}$$

$$\frac{\Gamma \to F\langle {}^{-}A' \rangle}{\Gamma, {}^{i}_{n}\neg F\langle {}^{-}A' \rangle \to}\ (\neg \to) \qquad \frac{{}^{l+1}[\Gamma] \to {}^{l+1}[F] \quad \Gamma, G\langle {}^{-}A' \rangle \to {}^{+}A}{\Gamma, F \,{}^{i}_{n}\!\supset G\langle {}^{-}A' \rangle \to {}^{+}A}\ (\supset \to)$$

$$\frac{\Gamma, F\langle {}^{-}A' \rangle \to {}^{+}A}{\Gamma, F\langle {}^{-}A' \rangle \,{}^{i}_{n}\!\wedge G \to {}^{+}A}\ (\wedge_1 \to) \qquad \frac{\Gamma, G\langle {}^{-}A' \rangle \to {}^{+}A}{\Gamma, F \,{}^{i}_{n}\!\wedge G\langle {}^{-}A' \rangle \to {}^{+}A}\ (\wedge_2 \to)$$

$$\frac{\Gamma, F\langle {}^{-}A' \rangle \to {}^{+}A \quad {}^{l+1}[\Gamma], {}^{l+1}[G] \to {}^{l+1}[{}^{+}A]}{\Gamma, F\langle {}^{-}A' \rangle \,{}^{i}_{n}\!\vee G \to {}^{+}A}\ (\vee_1 \to)$$

$$\frac{{}^{l+1}[\Gamma], {}^{l+1}[F] \to {}^{l+1}[A] \quad \Gamma, G\langle {}^{-}A' \rangle \to {}^{+}A}{\Gamma, F \,{}^{i}_{n}\!\vee G\langle {}^{-}A' \rangle \to {}^{+}A}\ (\vee_2 \to)$$

$$\mathit{Contraction\ rules:}$$

$$\frac{\Gamma, {}^{l+1}[F], F\rangle \to G}{\Gamma, F \to G}\ (Con \to) \qquad \frac{\Gamma \to {}^{l+1}[A]}{\Gamma \to A}\ (\to Con)$$

$$\mathit{Quantifier\ rules:}$$

$$\frac{\Gamma \to {}^{l+1}[G]|^{{}^{i}_{n}x}_{(l+1)+{}^{i}_{n}\overline{x}}}{\Gamma \to {}^{i}_{n}\forall^{i}_{n}xG}\ (\to \forall) \qquad \frac{\Gamma, {}^{l+1}[F]|^{{}^{i}_{n}x}_{(l+1)+{}^{i}_{n}\underline{x}} \to G}{\Gamma, {}^{i}_{n}\forall^{i}_{n}xF \to G}\ (\forall \to)$$

$$\frac{\Gamma, {}^{l+1}[F]|^{{}^{i}_{n}x}_{(l+1)+{}^{i}_{n}\overline{x}} \to G}{\Gamma, {}^{i}_{n}\exists^{i}_{n}xF \to G}\ (\exists \to) \qquad \frac{\Gamma \to {}^{l+1}[G]|^{{}^{i}_{n}x}_{(l+1)+{}^{i}_{n}\underline{x}}}{\Gamma \to {}^{i}_{n}\exists^{i}_{n}xG}\ (\to \exists)$$

Γ is a multiset of formulas, F and G are formulas, A and A' are atomic formulas; at that, $A \approx A'$ w.r.t. an mgsu. In the succedents of the sequents from the $(\forall \to)$, $(\exists \to)$, $(Con \to)$, and premise splitting rules, G and A can be absent. ${}^{(l+1)+i}_{\quad\ \ n}\underline{x}$ and ${}^{(l+1)+i}_{\quad\ \ n}\overline{x}$ are (new) dummy and parameter respectively. The operation ${}^{l+1}[\cdot]$ replaces all the left upper indexes in F, G, and Γ by new indexes by the procedure given in [1] and produces new "copies" of formulas, where l is the largest left upper index in an inference tree constructed to the time of application of a corresponding rule. $|^{{}^{i}_{n}x}_{(l+1)+{}^{i}_{n}\overline{x}}$ ($|^{{}^{i}_{n}x}_{(l+1)+{}^{i}_{n}\underline{x}}$) denotes the replacing of ${}^{i}_{n}x$ by ${}^{(l+1)+i}_{\quad\ \ n}\overline{x}$ (by ${}^{(l+1)+i}_{\quad\ \ n}\underline{x}$).

Figure 1. Sequent calculus $\mathbf{LJ}^{*}$ for intuitionistic logic

3 Main result

Proof search in $\mathbf{LJ}^*$ has the form a tree called an *inference tree* growing in the Kander style [3] from bottom to top in attempting to produce a *latent proof tree*, being a tree, all leaves of which are of the form $\Gamma, A' \to A$, where A and A' are atomic formulas and $A' \approx A$. It consists of three steps: firstly we try to apply (in any order) quantifier rules and, maybe, several contraction rules as long as they are applicable, then we try to apply goal splitting rule as long as they are applicable, and after this we try to apply premise splitting rules ("driven" by an atomic formula from the succedent of a sequent under consideration).

Theorem. *For a closed formula F, the sequent $\to F$ is deducible in* ***LJ*** *(i.e. F is intuitionistically valid) if and only if an inference tree Tr for the starting sequent $\to {}^1[_\nu F]$ can be constructed in the* ***LJ***** *calculus and a substitution σ of terms without dummies for all the free dummies from Tr can be selected in such a way that the following conditions take place: (i) Tr is a latent proof tree for $\to {}^1[_\nu F]$; (ii) the application of σ to all the leaves of Tr converts these leaves into axioms; (iii) σ is admissible for Tr, (iv) Tr is compatible with σ w.r.t.* ***LJ*****.

In conclusion, the author hopes that ideas presented in this work will be useful in constructing intuitionistic first-order provers.

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Refutation Search and Literal Trees Calculi

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Abstract

Special calculi which can be considered as extensions of the SLD and input resolutions on the case of sets of abitrary clauses are constructed for doing refutation in the form of tree-like structures. The results on their validity and completeness are given.

Keywords: classical first-order clausal logic, clause, literal tree, SLD resolution, input resolution, refutation, satisfiability.

1 Introduction

Usually, many intelligent systems, for example, logical programming systems, contain different versions of the so-called SLD resolution or input resolution as their logical engines providing a complete refutation technique for sets of Horn clauses. But they are not complete for sets of arbitrary clauses. In this connection, the following question arises: How should we modify the SLD or input resolution in order to get their extensions being complete in the general case?

There are results that gives methods for constructing such refutation extensions of the SLD and input resolutions (see, for example, [1, 2]) and they can be interpreted as complete refutation search methods in the *linear format* [3]. This paper contains another answer on the above-asked question for making *refutation search in the form of tree-like structures* being a well-formed expressions of certain calculi.

2 Calculi of Literal Trees

Classical first-order logic without equality is considered. The problem of a refutation of a set of clauses is solved. In this connection, all the notions of the Robinson resolution technique are assumed to be known (see, for example, [4]). Remind that a *literal* is an atomic formula or its negation.

For refutation search, i.e. for the establishing the insufficiency of an *input set* M_I being a set of original (input) clauses, two special calculi of so-called literal trees are constructed. They are denoted by **LT** and $\mathbf{LT}^\sharp$, in which literal trees "grow from top to bottom".

A *literal tree* is a usual tree, the root of which is labeled by M_I and all the other nodes are labeled by literals. Additionally, we assume that the symbol $\sharp$ can be used as a label for any leaf of a literal tree.

A tree without any nodes is denoted by $\triangle$.

A literal tree is called *closed*, if it is $\triangle$ or all the labels of all its leaves are $\sharp$.

A *variant* of a literal tree Tr is called a tree, the nodes of which are variants (in the usual sense) of literals of Tr.

If Tr is a literal tree and σ a substitution, then $Tr \cdot \sigma$ is the result of the application of σ to all the labels of the nodes of Tr ($\sharp \cdot \sigma = \sharp$).

Let M_I be an input set of clauses and a clause $L_1 \vee \ldots \vee L_n$ belongs to M_I, where $L_1, \ldots, L_n$ are literals. Then the tree consisting of the root labeled by M_I and n its successors labeled by $L_1, \ldots, L_n$ is called an *input (literal) tree* for M_I w.r.t. $L_1 \vee \ldots \vee L_n$.

New trees are produced by the inference rules given below. At that, any *inference* in a literal trees calculus is a sequence of literal trees $Tr_1, \ldots, Tr_m$, in which Tr_1 is an input tree and for the each $1 < j \leq n$, Tr_j is deduced from a variant of Tr_{j-1} by one of the inference rules.

Let $C \in M_I, Tr_1, \ldots, Tr_m$ be an inference in a literal trees calculus, in which Tr_1 is an input tree for M_I w.r.t. C and Tr_m is a closed tree. Then $Tr_1, \ldots, Tr_m$ is called a *refutation of M_I w.r.t. C* in this calculus.

The calculus **LT** contain the below given IC and CL rules.

Input clause extension rule (IC rule). Let M_I be an input set of clauses and $L_1 \vee \ldots \vee L_n \in M_I$ $(n \geq 1)$. Let Tr be a variant a tree not having common variables with $L_1 \vee \ldots \vee L_n$ and E be the most right leaf of Tr distinguishing from $\sharp$. Suppose that for some i $(1 \leq i \leq n)$ there exists the most general unifier σ of the set $\{L_i, \overline{E}\}$, where $\overline{E}$ denotes the complementary to E. Let us construct the literal tree Tr' in the following way: (1) if $n > 1$, then Tr' is constructed from Tr by adding $n-1$ nodes to the node with E as its successors and assigning $L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n$ as their labels and (2) if $n = 1$, then Tr' is constructed from Tr by adding the unique node to the node with E as its successor and assigning $\sharp$ as its label. Then $Tr' \cdot \sigma$ *is said to be deducable from* Tr *and* $L_1 \vee \ldots \vee L_n$ *by the IC rule.*

Contrary Literals rule (CL rule). Let Tr be a literal tree and Br its the most right branch with the leaf labeled by a literal L distinguished from $\sharp$. Suppose Br contains a node with such a literal E that there exists the most general unifier σ of the set $\{L, \overline{E}\}$, where $\overline{E}$ denotes the complementary to E. If Tr' is constructed from Tr by adding the unique node to the leaf with L as its successor and assigning $\sharp$ as its label, then $Tr' \cdot \sigma$ *is said to be deducable from* Tr *by the CL rule.*

Theorem 1. *If M_I is an input set of clauses, $C \in M_I$, and $M_I \backslash \{C\}$ is a satisfied set, then M_I is an unsatisfiable set if and only if in the* ***LT*** *calculus there exists a refutation of M_I w.r.t. C.*

Refutation search in **LT** leads to producing successively increasing trees, which is not good in the case of implementing **LT**. The next rule is intended for improving this situation.

Chain Scratching rule (CS rule). Let Tr be a literal tree and Br its branch with a leaf L labeled by $\sharp$. Suppose Ch denotes such a maximal chain in Br that Ch contains L and each node of Ch except L has only one successor. Then a tree Tr' constructed by deleting Ch in Tr is called a *tree deduced from* Tr *by the* CS *rule.*

By $\mathbf{LT}^\sharp$ denote the **LT** calculus, to which the *CS* rule is added. We require that in $\mathbf{LT}^\sharp$, the CS *rule always is applied* after any applications of the *IC* and *CL* rules.

Theorem 2. *If M_I is an input set of clauses, $C \in M_I$, and $M_I \backslash \{C\}$*

is a satisfied set, then M_I is an unsatisfiable set if and only if in the $\boldsymbol{LT}^\sharp$ *calculus there exists such a refutation $Tr_1, \ldots, Tr_m$ of M_I w.r.t. C that Tr_m is $\triangle$.*

3 Literal trees calculi and complete tree-like extensions of SLD and input resolutions

The SLD and input resolutions are usually used in many intelligent systems as their logical engines. They are complete methods for an input set of Horn clauses and are not complete in the general case.

Comparing the SLD and input resolutions with the **LT** and $\mathbf{LT}^\sharp$ calculi and using Theorems 1 and 2, we can get the following result.

Corollary. *The* ***LT*** *and* $\boldsymbol{LT}^\sharp$ *calculi can be considered as complete tree-like extensions of the SLD and input resolutions on the case of an input set of arbitrary clauses.*

Corollary demonstrates that the calculi **LT** and $\mathbf{LT}^\sharp$ can be "guidelines" for making complete tree-like extensions of SLD and input resolutions on the case of an input set of arbitrary clauses by adding to them the simple *CL* and *CS* rules. That is we have an uncomplicated way for increasing deductive capabilities of one or other intelligent system with a logical engine based on the SLD or input resolution.

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On Recursive Derivates of k-ary Operations

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Abstract

We present several results about recursive derivates of k-ary operations defined on a finite set Q. They are generalizations of some binary cases given by Larionova-Cojocaru and Syrbu [5]. Also, we present several experimental results about recursive differentiability of ternary quasigroups of order 4.

Keywords: recursively differentiable quasigroups, orthogonality

1 Introduction

Let Q be a nonempty set and let k be a positive integer. We will use (x_1^k) to denote the k-tuple $(x_1, \ldots, x_k) \in Q^k$. A *$k-$ary operation* f on the set Q is a mapping $f : Q^k \to Q$ defined by $f : (x_1^k) \to x_{k+1}$, for which we write $f(x_1^k) = x_{k+1}$. A k-ary groupoid ($k \geq 1$) is an algebra (Q, f) on a nonempty set Q as its universe and with one k-ary operation f. A k-ary groupoid (Q, f) is called a *k-ary quasigroup* (of order $|Q| = q$) if any k of the elements $a_1, a_2, \ldots, a_{k+1} \in Q$, satisfying the equality $f(a_1^k) = a_{k+1}$, uniquely specifies the remaining one.

The $k-$ary operations $f_1, f_2, \ldots, f_d, 1 \leq d \leq k$, defined on a set Q are **orthogonal** if the system $\{f_i(x_1^k) = a_i\}_{i=1}^d$ has exactly q^{k-d} solutions for any $a_1, \ldots, a_d \in Q$, where $q = |Q|$ [2]. There is an one-to-one correspondence between the set of all k-tuples of orthogonal k-ary operations $< f_1, f_2, \ldots, f_k >$ defined on a set Q and the set of all permutations $\theta : Q^k \to Q^k$ ([2]), given by

$$\theta(x_1^k) \to (f_1(x_1^k), f_2(x_1^k), \ldots, f_d(x_1^k)).$$

The k-ary operation I_j, $1 \le j \le k$, defined on Q with $I_j(x_1^k) = x_j$ is called the *j-th selector* or the *j-th projection*.

A system $\Sigma = \{f_1, f_2, \dots, f_s\}_{s \ge k}$ of k-ary operations is called **orthogonal**, if every k operations of Σ are orthogonal. A system $\Sigma = \{f_1, f_2, \dots, f_r\}, r \ge 1$ of distinct $k-$ary operations defined on a set Q is called **strong orthogonal** if the system $\{I_1, \dots, I_k, f_1, f_2, \dots, f_r\}$ is orthogonal, where each $I_j, 1 \le j \le k$, is $j-$th selector. It follows that each operation of a strong orthogonal system, which is not a selector, is a k-ary quasigroup operation.

A code $C \subseteq Q^n$ is called a **complete k-recursive code** if there exists a function $f : Q^k \to Q$ $(1 \le k \le n)$ such that every code word $(u_0, \dots, u_{n-1}) \in C$ satisfies the conditions $u_{i+k} = f(u_i^{i+k-1})$ for every $i = 0, 1, \dots, n-k-1$, where $u_0, \dots, u_{k-1} \in Q$. It is denoted by $C(n, f)$.

$C(n, f)$ can be represented by

$$C(n, f) = \{(x_1^k, f^{(0)}(x_1^k), \dots, f^{(n-k-1)}(x_1^k)) : (x_1^k) \in Q^k\}$$

where $f^{(0)} = f^{(0)}(x_1^k) = f(x_1^k)$,
$f^{(1)} = f^{(1)}(x_1^k) = f(x_2^k, f^{(0)})$
$\dots$
$f^{(k-1)} = f^{(k-1)}(x_1^k) = f(x_k, f^{(0)}, \dots, f^{(k-2)})$
$f^{(i+k)} = f^{(i+k)}(x_1^k) = f(f^{(i)}, \dots, f^{(i+k-1)})$ for $i \ge 0$
are **recursive derivatives** of f. The general form of the recursive derivatives for any k-ary operation f is given in [4], and $f^{(n)} = f\theta^n$, where $\theta : Q^k \to Q^k, \theta(x_1^k) = (x_2^k, f(x_1^k))$.

A k-quasigroup (Q, f) is called **recursively t-differentiable** if all its recursive derivatives $f^{(0)}, \dots, f^{(t)}$ are k-ary quasigroup operations [3]. A k-quasigroup (Q, f) is called **t-stable** if the system of all recursive derivatives $f^{(0)} \dots, f^{(t)}$ of f is an orthogonal system of k-ary quasigroup operations, i.e. $C(k+t+1, f)$ is an MDS code [3]. A k-ary quasigroup (Q, f) is called **strongly recursively t-differentiable** if it is recursively t-differentiable and $f^{(t+1)} = I_1$ (introduced for binary case in [1]). A k-ary quasigroup (Q, f) is strongly recursively 0-differentiable if $f^{(1)} = I_1$.

2 Main results

The following results are generalisation of binary cases for recursive derivates from [5].

Proposition 1. *Let (Q, f) be a k-ary groupoid. For every $(x_1^k) \in Q^k$ the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k)), \forall n \in N$$

Proposition 2. *Let (Q, f) be a k-ary groupoid. For every $(x_1^k) \in Q^k$ and for every $j = k-1, \ldots, n-1$, where $n \geq k$, the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-j-1)}(f^{(j-k+1)}(x_1^k), \ldots, f^{(j)}(x_1^k))$$

Proposition 3. *If two k-ary groupoids (Q_1, f) and (Q_2, g) are isomorphic, then their recursive derivatives $(Q_1, f^{(n)})$ and $(Q_2, g^{(n)})$ are isomorphic too, for every $n \geq 1$.*

Proposition 4. *If (Q, f) is a k-ary groupoid, then $Aut(Q, f)$ is a subgroup of $Aut(Q, f^{(n)})$, for every $n \geq 1$.*

3 Experimental results for ternary quasigroups of order 4

By experiments, we obtained the following results:

- there are 96 recursively 1-differentiable ternary quasigroups of order 4, and all are 1-stable
- there are no recursively t-differentiable ternary quasigroups of order 4, for $t \geq 2$,
- there are 64 strongly recursively 0-differentiable ternary quasigroups of order 4,
- there are 8 strongly recursively 1-differentiable ternary quasigroups of order 4.

Bellow is an example of strongly recursively 1-differentiable and 1-stable ternary quasigroups of order 4.

$$\{\{\{1,2,3,4\},\{3,4,1,2\},\{4,3,2,1\},\{2,1,4,3\}\},\{\{2,1,4,3\},\{4,3,2,1\},\{3,4,1,2\},\{1,2,3,4\}\},$$
$$\{\{3,4,1,2\},\{1,2,3,4\},\{2,1,4,3\},\{4,3,2,1\}\},\{\{4,3,2,1\},\{2,1,4,3\},\{1,2,3,4\},\{3,4,1,2\}\}\}$$

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Non-commutative finite rings with several mutually associative multiplication operations

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Abstract

Some properties of non-commutative finite rings of four-dimension vectors with several mutually associative multiplication operations are presented.

Keywords: ring, Galois field, vector, local left unit element, bi-side unit element, associative law.

1 Introduction

Finite non-commutative rings are of interest for designing public-key cryptoschemes based on the discrete logarithm problem in hidden commutative subgroup [1, 2, 3].

Suppose $\mathbf{e}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be some formal basis vectors and $a, b, c, d \in GF(p)$, where $p \geq 3$ is a prime number, are coordinates. The vectors are denoted as $a\mathbf{e} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ or as (a, b, c, d). The terms $\tau\mathbf{v}$, where $\tau \in GF(p^d)$ and $\mathbf{v} \in \{\mathbf{e}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, are called components of the vector.

The addition of two vectors (a, b, c, d) and (x, y, z, v) is defined via addition of the corresponding coordinates accordingly to the following formula $(a, b, c, d) + (x, y, z, v) = (a + x, b + y, c + z, d + v)$.

The multiplication of two vectors $a\mathbf{e}+b\mathbf{i}+c\mathbf{j}+z\mathbf{k}$ and $x\mathbf{e}+y\mathbf{i}+z\mathbf{j}+v\mathbf{k}$ is defined by the following formula

$$(a\mathbf{e} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \circ (x\mathbf{e} + y\mathbf{i} + z\mathbf{j} + v\mathbf{k}) =$$
$$= ax\mathbf{e}\circ\mathbf{e}+bx\mathbf{i}\circ\mathbf{e}+cx\mathbf{j}\circ\mathbf{e}+dx\mathbf{k}\circ\ldots\circ\mathbf{j}+av\mathbf{e}\circ\mathbf{k}+bv\mathbf{i}\circ\mathbf{k}+cv\mathbf{j}\circ\mathbf{k}+dv\mathbf{k}\circ\mathbf{k},$$

Table 1. The basis-vector multiplication table

$\circ$	$\vec{e}$	$\vec{\imath}$	$\vec{\jmath}$	$\vec{k}$
$\vec{e}$	$\mathbf{e}$	$\mu\mathbf{k}$	$\mu\mathbf{e}$	$\mathbf{k}$
$\vec{\imath}$	$\tau\mathbf{j}$	$\mathbf{i}$	$\mathbf{j}$	$\tau\mathbf{i}$
$\vec{\jmath}$	$\mathbf{j}$	$\mu\mathbf{i}$	$\mu\mathbf{j}$	$\mathbf{i}$
$\vec{k}$	$\tau\mathbf{e}$	$\mathbf{k}$	$\mathbf{e}$	$\tau\mathbf{k}$

where $\circ$ denotes the vector multiplication operation and each product of two basis vectors is to be replaced by some basis vector or by a one-component vector in accordance with the basis-vector multiplication table (BVMT) defining associative and non-commutative multiplication. In this paper there is introduced a novel BVMT that defines parameterized multiplication operation different modification of which are mutually associative. The proposed BVMT is shown in Table 1, where $\mu \in GF(p)$ and $\tau \in GF(p)$ are structural coeficients.

2 Properties of the introduced ring

Statement 1. Suppose $\circ$ and $\star$ are two arbitrary modifications of the vector multiplication operation, which correspond to different pair of structural coefficients (μ_1, τ_1) and $(\mu_2, \tau_2) \neq (\mu_1, \tau_1)$. Then for arbitrary three vectors A, B, and C it holds the following formula $(A \circ B) \star C = A \circ (B \star C)$.

Statement 2. The vector $E = \left(\frac{1}{1-\mu\tau}, \frac{1}{1-\mu\tau}, \frac{\tau}{\mu\tau-1}, \frac{\mu}{\mu\tau-1}\right)$ is the (global) unity element of the considered ring, i.e. for arbitrary vector V it holds $V \circ E = E \circ V = V$.

Statement 3. Vectors $V = (a, b, c, d)$, where $ab \neq cd$, are invertible.

Statement 4. The order Ω of the multiplicative group of the considered ring is equal to $\Omega = p(p-1)(p^2-1)$.

Statement 5. For an arbitrary vector $N = (a, b, c, d)$ such that

$ab = cd$ and $a\tau + c \neq 0$ each of the vectors

$$E_l = \left(x, \frac{c}{a\tau + c} - \frac{a + c\mu}{a\tau + c} z, z, \frac{a}{a\tau + c} - \frac{a + c\mu}{a\tau + c} x \right),$$

where $x, y \in GF(p)$ acts as the left local unity element for all elements N^i, where $i > 0$ is an arbitrary integer, i.e. it holds $E_l \circ N^i = N^i$.

Statement 6. For an arbitrary vector $N = (a, b, c, d)$ such that $ab = cd$ and $a\mu + d \neq 0$, each of the vectors

$$E_r = \left(x, \frac{d}{a\mu + d} - \frac{a + d\tau}{a\mu + d} w, \frac{a}{a\mu + d} - \frac{a + d\tau}{a\mu + d} x, w \right),$$

where $x, w \in GF(p)$ acts as the right local unity element for all elements N^i, where $i > 0$ is an arbitrary integer, i.e. it holds $N^i \circ E_r = N^i$.

Statement 7. Suppose $N = (a, b, c, d)$ be a non-invertible vector, i.e. $ab = cd$. Then the sequence $N, N^2, ..., N^i, ...$, where $i = 1, 2, ...$, is periodic and for some integer ω we have $N^\omega = E'$, where E' is the local unity element such that $N \circ E' = E' \circ N = N$.

Statement 8. The local bi-side unity element E' is described with the following formula $E' = \left(x_0, \frac{d}{a\mu+d} - \frac{a+d\tau}{a\mu+d} \cdot \frac{d}{a} x_0, \frac{d}{a\mu+d} - \frac{a+d\tau}{a\mu+d} x_0, \frac{d}{a} x_0 \right)$, where $x_0 = \frac{a^2}{ca\mu+cd+a^2+ad\tau}$.

Statements 5 to 7 show that the set of non-invertible vectors includes different cyclic groups with different local unity elements.

There exist homomorphisms of the multiplicative group of the considered ring into the set of non-invertible vectors. Suppose $N = (a, b, c, d)$ be a non-invertible vector such that $N^\omega = E'$ and V_1, V_2 are two invertible vectors. The function $\varphi(V) = N^{\omega - i} \circ V \circ N^i$ defines a homomorphic mapping: $\varphi(V_1 \circ V_2) = \varphi(V_1) \circ \varphi(V_2)$. The function $\varphi(W) = N^{\omega - i} \circ W \circ N^i$ defines a homomorphic mapping of the considered ring into the set of its non-invertible vectors: $\varphi(W_1 \circ W_2) = \varphi(W_1) \circ \varphi(W_2)$; $\varphi(W_1 + W_2) = \varphi(W_1) + \varphi(W_2)$, where W_1 and W_2 are two arbitrary vectors of the ring. Selecting different values i and different non-invertible vectors N it is possible to define a variety of homomorphic maps.

Like in papers [2, 3], one can construct public-key crypto-schemes using the homomorphisms in the considered finite ring of four dimension vectors.

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Program-oriented Logics of Non-Deterministic Quasiary Predicates

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Abstract

Program-oriented logics defined for classes of quasiary predicates are studied. Such predicates are partial predicates over partial states (partial assignments) of variables. Conventional n-ary predicates can be considered as a special case of quasiary predicates. We define extended first-order logics of quasiary non-deterministic predicates. A special consequence relation, adequate for such logics, is introduced and its semantic properties are studied. Obtained results are used to prove logic validity and completeness.

Keywords: first-order logic, quasiary predicate, partial predicate, non-deterministic predicate.

1 Introduction

Logics of quasiary predicates can be considered as a natural generalization of classical predicate logic. The latter is based upon total n-ary predicates which represent fixed and static properties of subject domain. Though classical logics and its various extensions are wildly used in computer science some restrictions of such logics should be mentioned. For example, in computer science partial and non-deterministic predicates over complex data structures often appear. Therefore there is a need to construct such logical systems that better reflect abovementioned features. One of specific features for computer science is quasiarity of predicates. Such predicates are partial predicates defined over partial states (partial assignments) of variables and, consequently, they do not have fixed arity. Conventional n-ary predicates can be considered as a special case of quasiary predicates.

In our previous works [1, 2] we primarily investigated the class of partial deterministic predicates and constructed corresponding logics. Here we aim to construct a logic of non-deterministic quasiary predicates. The logic construction consists of several phases: first, we construct predicate algebras, terms of which specify the language of a logic; then we define interpretation mappings and a consequence relation; at last, we construct a calculus for the defined logic.

2 Algebras of non-deterministic quasiary predicates

Let V and A be nonempty sets of *variables* (*names*) and *basic values* respectively. Given V and A, the class ${}^{V}\!A$ of nominative sets is defined as the class of all partial mappings from V to A, thus, ${}^{V}\!A = V \xrightarrow{p} A$. Intuitively, nominative sets represent states of variables.

The main operation for nominative sets is a total unary parametric *renomination* $r^{v_1,\ldots,v_n}_{x_1,\ldots,x_n} : {}^{V}\!A \xrightarrow{t} {}^{V}\!A$ where $v_1, ..., v_n, x_1, ..., x_n \in V$, $v_1, ..., v_n$ are distinct names, $n \geq 0$. Informally, given d this operation yields a new nominative set changing the values of $v_1, ..., v_n$ to the values of $x_1, ..., x_n$ respectively.

Let $PrR^{V}_{A} = {}^{V}\!A \xrightarrow{r} Bool$ be the set of all non-deterministic (relational) predicates over ${}^{V}\!A$. Such predicates are called *non-deterministic quasiary predicates.* Note that non-determinism in logic was intensively studied, but primarily for propositional level, see, for example, [3].

For $p \in PrR^{V}_{A}$ the truth and falsity domains of p are denoted $T(p)$ and $F(p)$ respectively.

Operations over PrR^{V}_{A} are called *compositions.* Basic compositions for quasiary predicates are disjunction $\vee$, negation $\neg$, renomination $R^{v_1,\ldots,v_n}_{x_1,\ldots,x_n}$, and existential quantification $\exists x$. We extend them with null-ary composition εx called variable unassignment predicate. Thus, the extended set $CE(V)$ of first-order compositions is $\{\vee, \neg, R^{v_1,\ldots,v_n}_{x_1,\ldots,x_n}, \exists x, \varepsilon z\}$.

Please note that the compositions are similar to strong Kleene's connectives and quantifiers.

A pair $AQE(V, A) =< PrR^V_A; CE(V) >$ is called *a first-order extended algebra of non-deterministic quasiary predicates.*

We investigate the main semantic properties of such algebras, focusing on properties of renominations and quantifiers.

3 Logic of non-deterministic quasiary predicates

Algebras $AQE(V, A)$ (for various A) form a semantic base for the constructed first-order extended quasiary predicate logic L^{QE}. The set of terms (formulas) specifies the logic language. Formula interpretations are defined in a traditional way.

Usually, for logics of quasiary predicates an irrefutability consequence relation is defined [1, 2]: a set of formulas Δ is a consequence of a set of formulas Γ in an interpretation J, if $\bigcap\limits_{\Phi\in\Gamma} T(\Phi_J) \cap \bigcap\limits_{\Psi\in\Delta} F(\Psi_J) = \emptyset$.

For the class of non-deterministic predicates this consequence relation is very poor. Therefore we introduce another consequence relation which arises naturally in computer science [4]: Δ is a *TF-consequence* of Γ in an interpretation J, if

$\bigcap\limits_{\Phi\in\Gamma} T(\Phi_J) \subseteq \bigcup\limits_{\Psi\in\Delta} T(\Psi_J)$ and $\bigcup\limits_{\Phi\in\Gamma} F(\Phi_J) \supseteq \bigcap\limits_{\Psi\in\Delta} F(\Psi_J)$.

Properties of this consequence relation are investigated. It is proved that this relation is paraconsistent, paracomplete, and paranormal.

4 Sequent calculus for logic of non-deterministic quasiary predicates

Sequent rules are derived from the properties of the consequence relation, but additionally we should introduce special sequent closedness conditions that take into consideration unassigned variables. Obtained calculus is called QE-calculus. We prove the main result of the paper: *QE-calculus is sound and complete*.

5 Conclusion

In this paper we have studied program-oriented logics defined for classes of non-deterministic quasiary predicates. For such logics we have constructed a special sequent calculus and proved its soundness and completeness.

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Asymptotics of isoperimetric functions of groups

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Abstract

The minimal non-decreasing function $f\colon \mathbf{N} \to \mathbf{N}$ such that every word w vanishing in a group $G = \langle A \mid R \rangle$ and having length $||w|| \leq n$ is freely equal to a product of at most $f(n)$ conjugates of relators from R, is called the *isoperimetric or Dehn function* of the presentation $G = \langle A \mid R \rangle$. By van Kampen Lemma, $f(n)$ is equal to the maximal area of minimal diagrams Δ with $||\partial\Delta|| \leq n$. For *finitely presented* groups (i.e., both sets A and R are finite) isoperimetric functions are usually taken up to equivalence to get rid of the dependence on a finite presentation for G. To introduce this equivalence $\sim$, we write $f \preceq g$ if there is a positive integer c such that $f(n) \leq cg(cn) + cn$ for any $n \in \mathbf{N}$. Two non-decreasing functions f and g on $\mathbf{N}$ are called equivalent if $f \preceq g$ and $g \preceq f$. Almost complete description of rapidly increasing isoperimetric functions (at least biquadratic) and the connection to the computation complexity of the word problem in groups can be found in [9], [1] and [7].

In fact, a subquadratic isoperimetric functions of finitely presented group G is linear up to equivalence, and the isoperimetric function of G is equaivalent to linear function if and only if the group G is hyperbolic (see [3, 6.8.M], [5], [2]). The speaker will discuss the asymptotic behavior of isoperimetric functions close to quadratic ones. In particular, he will pay attention to his recent results.

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Factorizations in the matrix ring and in its subrings of the block matrices

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Abstract

The factorizations in the ring of the matrices over an integral domain of finitely generated principal ideals are described. We establish the conditions under which the factorizations of the block matrices in the matrix ring up to the association are the factorizations in its subrings of the block triangular and the block diagonal matrices.

Keywords: matrix ring, block matrix, factorization.

1 Introduction

Let R be an integral domain of finitely generated principal ideals. We will denote the ring of $n \times n$ matrices over R by $M(n, R)$, the subring of the block upper triangular matrices $T = \mathrm{triang}(T_{11}, \ldots, T_{kk}) = [T_{ij}]_1^k, T_{ij} = 0$ if $i > j$, $T_{ii} \in M(n_i, R)$ by $BT(n_1, \ldots, n_k, R)$, the subring of the block diagonal matrices $D = \mathrm{diag}(D_{11}, \ldots, D_{kk}),\ D_{ii} \in M(n_i, R)$ by $BD(n_1, \ldots, n_k, R)$.

We describe the factorizations of the matrices in the ring $M(n, R)$ and in its subrings $BT(n_1, \ldots, n_k, R)$ and $BD(n_1, \ldots, n_k, R)$. We establish the conditions under which the factorization of the matrix $T \in BT(n_1, \ldots, n_k, R)$ in the ring $M(n, R)$ up to the association is the factorization in the ring $BT(n_1, \ldots, n_k, R)$. There is also given a uniqueness criterion up to the association of such factorizations. In similar fashion, for the block diagonal matrices D from the ring $BD(n_1, \ldots, n_k, R)$.

We should note that the block matrices are used in different applied problems [1].

It is known the method of the factorization of the matrices over the polynomial rings, that is the decomposition of the polynomial matrices into a product of the regular factors [2], [3], [4]. The polynomial matrices and their factorizations have been used in the theory of matrix and differential equations, the theory of operator pencils and in other applied problems. In [5], it has been formulated the problem of the description up to the association of the matrix factorizations over the principal ideals rings and it has been established the conditions for uniqueness of such factorizations. The factorizations of the block matrices over the polynomial ring have been considered in [6] and over the principal ideals rings in [7].

2 Factorizations of the block matrices

Further, we will suppose that the matrix $T = \mathrm{triang}(T_{11}, \dots, T_{kk})$, $T_{ii} \in M(n_i, R)$, $i = 1, \dots, k$ from the ring $BT(n_1, \dots, n_k, R)$ is nonsingular.
Theorem 1. *Let the diagonal blocks T_{ii}, $i = 1, \dots, k$, of the matrix T have the factorizations of the form*

$$T_{ii} = B_{ii}C_{ii},\ B_{ii}, C_{ii} \in M(n_i, R) \quad i = 1, \dots, k. \tag{1}$$

Then there exists a unique up to the association the factorization of the matrix T in the ring $BT(n_1, \dots, n_k, R)$, that is

$$T = \mathrm{triang}(B_{11}, \dots, B_{kk})\mathrm{triang}(C_{11}, \dots, C_{kk}), \tag{2}$$

if and only if $(\det B_{ss}, \det C_{s+t,s+t}) = 1$ *for all* $s = 1, \dots, k-1$, $t = 1, \dots, k-s$.
Theorem 2. *Let the determinants of the diagonal blocks T_{ii} of the matrix T have the factorizations*

$$\det T_{ii} = \varphi_i\psi_i, \quad \varphi = \prod_{i=1}^{k} \varphi_i,\ \psi = \prod_{i=1}^{k} \psi_i.$$

1. If at least one of the following conditions hold:

(i) $(\prod_{i=1}^{s}\varphi_i, \psi_{s+1}) = 1$, and $((\varphi,\psi), d_{n-1}^{T}) = 1$, $s = 1,\dots,k-1$,

(ii) $(\det T_{ii}, (\varphi,\psi)) = 1$, $i = 1,\dots,k-1$,

there exists the factorization $T = BC$ *of the matrix* T *in the ring* $M(n,R)$, *that is* $B, C \in M(n,R)$, $\det B = \varphi$, $\det C = \psi$. *Each such factorization of the matrix* T *in the ring* $M(n,R)$ *is associated to the factorization* $T = \widetilde{B}\widetilde{C}$ *of the matrix* T *in the ring* $BT(n_1,\dots,n_k,R)$, where $\widetilde{B} = BV = \operatorname{triang}(\widetilde{B}_{11},\dots,\widetilde{B}_{kk})$, $\widetilde{C} = V^{-1}C = \operatorname{triang}(\widetilde{C}_{11},\dots,\widetilde{C}_{kk})$, $V \in GL(n,R)$, $\det \widetilde{B}_{ii} = \varphi_i$, $\det \widetilde{C}_{ii} = \psi_i$, $i = 1,\dots,k$.

2. The matrix T *has the unique up to the association factorization* $T = \operatorname{triang}(B_{11},\dots,B_{kk})\operatorname{triang}(C_{11},\dots,C_{kk})$, $\det B_{ii} = \varphi_i, \det C_{ii} = \psi_i$ in the ring $BT(n_1,\dots,n_k,R)$ *if and only if the following conditions hold:*

(i) $((\varphi_i,\psi_i), d_{n_i-1}^{T_{ii}}) = 1$, $i = 1,\dots,k$,

(ii) $(\varphi_s, \psi_{s+t}) = 1$ $\quad s = 1,\dots,k-1$, $\quad t = 1,\dots,k-s$.

Theorem 3. *Let* $D \in BD(n_1,\dots,n_k,R)$, *that is* $D = \operatorname{diag}(D_{11},\dots,D_{kk})$, $D_{ii} \in M(n_i,R)$, $i = 1,\dots,k$, *be a nonsingular matrix. Let the determinants of its diagonal blocks* D_{ii} *have the factorizations:*

$$\det D_{ii} = \varphi_i\psi_i,\ \varphi = \prod_{i=1}^{k}\varphi_i,\ \psi = \prod_{i=1}^{k}\psi_i,\ i = 1,\dots,k.$$

If $((\det D_{ii}, \det D_{jj}), (\varphi,\psi)) = 1$, $i,j = 1,\dots,k$, $i \neq j$, *then there exists the factorization* $D = BC$, $B, C \in M(n,R)$, $\det B = \varphi$, $\det C = \psi$, *of the matrix* D *in the ring* $M(n,R)$. *Each of such factorization of the matrix* D *in the ring* $M(n,R)$ *is associated to the factorization* $D = \widetilde{B}\widetilde{C}$ *of the matrix* D *in the ring* $BD(n_1,\dots,n_k,R)$, *where* $\widetilde{B} = BV = \operatorname{diag}(\widetilde{B}_{11},\dots,\widetilde{B}_{kk})$, $\widetilde{C} = V^{-1}C = \operatorname{diag}(\widetilde{C}_{11},\dots,\widetilde{C}_{kk})$, $V \in GL(n,R)$, $\widetilde{B}_{ii}, \widetilde{C}_{ii} \in M(n_i,R)$, $\det \widetilde{B}_{ii} = \varphi_i$, $\det \widetilde{C}_{ii} = \psi_i$, $i = 1,\dots,k$.

The factorizations of the block matrices are constructed from the solutions of the system of the linear matrix equations.

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DCC on closed ideals in rings of continuous endomorphisms of LCA groups

Valeriu Popa

Abstract

We present here a description of LCA groups whose ring of continuous endomorphisms, endowed with the compact-open topology, satisfies DCC on closed right (respectively, left) ideals.

Keywords: LCA groups, rings of continuous endomorphisms, DCC on closed right ideals.

Let $\mathcal{L}$ be the class of LCA groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of continuous endomorphisms of X, taken with the compact-open topology. One may ask: For which groups $X \in \mathcal{L}$, the ring $E(X)$ satisfies the descending chain condition (DCC) on different types of closed ideals. In the following, we present some answers to this question. Let us fix some notations. Given $X \in \mathcal{L}$, we denote by $d(X)$ the maximal divisible subgroup of X, and by $m(X)$ the smallest closed subgroup K of X such that the quotient group X/K is torsion-free. If X is topologically torsion, $S(X)$ stands for the the set of those $p \in \mathbb{P}$ for which the topological p-primary component $X_p \neq \{0\}$. If $n \in \mathbb{N}$ and $A \subset X$, then $X[n] = \{x \in X \mid nx = 0\}$, $nX = \{nx \mid x \in X\}$, and $\overline{A}$ stands for the closure of A in X. The groups we shall use frequently are the cyclic groups $\mathbb{Z}(p^n)$ of order p^n, the rationals $\mathbb{Q}$ (all taken discrete), the character group $\mathbb{Q}^*$ of $\mathbb{Q}$, the p-adic numbers $\mathbb{Q}_p$, and the reals $\mathbb{R}$ (all with their usual topologies), where $n \in \mathbb{N}$ and $p \in \mathbb{P}$.

Theorem 1. Let X be a residual group in $\mathcal{L}$ such that the collection $\mathcal{E} = \{\overline{nE(X)} \mid n \text{ is a positive integer}\}$ has a minimal element with respect to set inclusion. Then X is a topological torsion group, and there exists a finite subset S of $S(X)$ such that the following conditions hold:

(i) For each $p \in S(X) \setminus S$, X_p is densely divisible and torsionfree;

(ii) For each $p \in S$, there exists an $n(p) \in \mathbb{N}$ such that $m(X_p) = X_p[p^{n(p)}]$ and $\overline{d(X_p)} = \overline{p^{n(p)} X_p}$.

Theorem 2. Let X be a group in $\mathcal{L}$ such that $E(X)$ satisfies DCC on topologically principal ideals. Then $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is a topological torsion group in $\mathcal{L}$ satisfying the following conditions:

(i) $S(Y)$ is finite;

(ii) for each $p \in S(Y)$, there exists $n(p) \in \mathbb{N}$ such that $m(Y_p) = Y[p^{n(p)}]$ and $\overline{d(Y_p)} = \overline{p^{n(p)} Y_p}$.

Theorem 3. Let $X \in \mathcal{L}$. The following statements are equivalent:

(i) $E(X)$ satisfies both ACC and DCC on closed right (respectively, left) ideals.

(ii) $E(X)$ satisfies DCC on closed right (respectively, left) ideals.

(iii) $E(X)$ satisfies DCC on topologically principal right (respectively, left) ideals.

(iv) $X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p \in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p \in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, where S_1, S_2 are finite subsets of $\mathbb{P}$, and $d, n, m,$ the $k(p)$'s, the $r_i(p)$'s and the $l(p)$'s are natural numbers.

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Local nearrings of order 243

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Abstract

All local nearrings on 2-generated groups of order 243 are classified.

Keywords: local nearring, 2-generated group, additive group.

In this paper the concept "nearring" means a left distributive nearring with an identity. Basic definitions and many results concerning nearrings can be for instance found in Pilz's book [1].

A nearring with identity is called local if the set of all its non-invertible elements is a subgroup of its additive group. The list of all local nearrings of order at most 31 can be extracted from the package "Sonata" [2] of the computer system algebra GAP [3].

We observe also that there exist 15 non-isomorphic groups of order $81 = 3^4$ from which 9 are the additive groups of local nearrings [4]. The following theorem describes 2-generated groups of order $243 = 3^5$ with this property. In particular, among 29 non-isomorphic 2-generated groups of this order only 10 are these additive groups.

Let G be i-th group of order n in the SmallGroups library of GAP. We write $IdGroup(G) = [n, i]$ and denote by C_n and D_n the cyclic and the dihedral group of order n. Furthermore, S_n means the symmetric group of degree n.

Theorem 1. *Let R be a local nearring of order 243, R^* the multiplicative group of R and L the additive subgroup of non-invertible elements of the additive group R^+ of R. If R^+ is 2-generated, then the pair of*

R^+ and L is one of the following:

[243, 2]	$(C_9 \times C_3) \rtimes C_9$	*and*	[81, 11]	$C_9 \times C_3 \times C_3$;
[243, 10]	$C_{27} \times C_9$	*and*	[81, 2]	$C_9 \times C_9$;
[243, 11]	$C_{27} \rtimes C_9$	*and*	[81, 2]	$C_9 \times C_9$;
[243, 12]	$(C_{27} \times C_3) \rtimes C_3$	*and*	[81, 11]	$C_9 \times C_3 \times C_3$;
[243, 13]	$((C_9 \times C_3) \rtimes C_3) \rtimes C_3$	*and*	[81, 15]	$C_3 \times C_3 \times C_3 \times C_3$;
[243, 14]	$(C_9 \times C_3) \rtimes C_9$	*and*	[81, 11]	$C_9 \times C_3 \times C_3$;
[243, 15]	$(C_9 \times C_3) \rtimes C_9$	*and*	[81, 11]	$C_9 \times C_3 \times C_3$;
[243, 16]	$(C_{27} \rtimes C_3) \rtimes C_3$	*and*	[81, 11]	$C_9 \times C_3 \times C_3$;
[243, 23]	$C_{81} \times C_3$	*and*	[81, 5]	$C_{27} \times C_3$;
[243, 24]	$C_{81} \rtimes C_3$	*and*	[81, 5]	$C_{27} \times C_3$.

The following table contains the numbers of all non-isomorphic local nearrings with given additive and multiplicative groups.

Theorem 2. *Let $n(G)$ be the number of all non-isomorphic local nearrings R whose additive group R^+ is isomorphic to the group G. If $n(R^*)$ is the number of these nearrings for which $IdGroup(R^*)$ is fixed, then the following holds.*

$IdGroup(G)$ $n(G)$	$IdGroup(R^*)$	$StructureDescription(R^*)$	$n(R^*)$
[243, 2] 119629	[162, 32]	$C_3 \times C_3 \times D_{18}$	12879
	[162, 33]	$C_3 \times C_9 \times S_3$	3351
	[162, 35]	$((C_3 \times C_3) \rtimes C_3) \times S_3$	87
	[162, 36]	$C_3 \times ((C_9 \rtimes C_3) \rtimes C_2)$	96472
	[162, 37]	$(C_9 \rtimes C_3) \times S_3$	6810
	[162, 51]	$C_3 \times C_3 \times C_3 \times S_3$	30
[243, 10] 119298	[162, 3]	$C_9 \times D_{18}$	477
	[162, 6]	$(C_9 \rtimes C_9) \rtimes C_2$	811
	[162, 23]	$C_{18} \times C_9$	21372
	[162, 25]	$C_2 \times (C_9 \rtimes C_9)$	96531
	[162, 30]	$C_2 \times ((C_9 \times C_3) \rtimes C_3)$	36
	[162, 31]	$C_2 \times (C_3.((C_3 \times C_3) \rtimes C_3) = = (C_3 \times C_3).(C_3 \times C_3))$	27
	[162, 47]	$C_{18} \times C_3 \times C_3$	6
	[162, 49]	$C_6 \times (C_9 \rtimes C_3)$	6
	[162, 50]	$C_2 \times ((C_9 \times C_3) \rtimes C_3)$	32

$IdGroup(G)$ $n(G)$	$IdGroup(R^*)$	$StructureDescription(R^*)$	$n(G)$
$[243, 11]$ 1288	$[162, 3]$	$C_9 \times D_{18}$	477
	$[162, 6]$	$(C_9 \rtimes C_9) \rtimes C_2$	811
$[243, 12]$ 29989	$[162, 33]$	$C_3 \times C_9 \times S_3$	10306
	$[162, 37]$	$(C_9 \rtimes C_3) \times S_3$	19683
$[243, 13]$ 139672	$[162, 32]$	$C_3 \times C_3 \times D_{18}$	2430
	$[162, 33]$	$C_3 \times C_9 \times S_3$	90
	$[162, 34]$	$C_3 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2)$	81811
	$[162, 35]$	$((C_3 \times C_3) \rtimes C_3) \times S_3$	6723
	$[162, 36]$	$C_3 \times ((C_9 \rtimes C_3) \rtimes C_2)$	32076
	$[162, 37]$	$(C_9 \rtimes C_3) \times S_3$	342
	$[162, 51]$	$C_3 \times C_3 \times C_3 \times S_3$	3321
	$[162, 52]$	$C_3 \times C_3 \times ((C_3 \times C_3) \rtimes C_2)$	12879
$[243, 14]$ 115020	$[162, 32]$	$C_3 \times C_3 \times D_{18}$	12393
	$[162, 33]$	$C_3 \times C_9 \times S_3$	3321
	$[162, 36]$	$C_3 \times ((C_9 \rtimes C_3) \rtimes C_2)$	92583
	$[162, 37]$	$(C_9 \rtimes C_3) \times S_3$	6723
$[243, 15]$ 117783	$[162, 32]$	$C_3 \times C_3 \times D_{18}$	11961
	$[162, 33]$	$C_3 \times C_9 \times S_3$	3432
	$[162, 34]$	$C_3 \times (((C_3 \times C_3) \rtimes C_3) \rtimes C_2)$	5832
	$[162, 36]$	$C_3 \times ((C_9 \rtimes C_3) \rtimes C_2)$	89127
	$[162, 37]$	$(C_9 \rtimes C_3) \times S_3$	6702
	$[162, 52]$	$C_3 \times C_3 \times ((C_3 \times C_3) \rtimes C_2)$	729
$[243, 16]$ 59049	$[162, 33]$	$C_3 \times C_9 \times S_3$	19683
	$[162, 37]$	$(C_9 \rtimes C_3) \times S_3$	39366
$[243, 23]$ 3349	$[162, 8]$	$C_{27} \times S_3$	28
	$[162, 26]$	$C_{54} \times C_3$	1134
	$[162, 27]$	$C_2 \times (C_{27} \rtimes C_3)$	2187
$[243, 24]$ 28	$[162, 8]$	$C_{27} \times S_3$	28

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On existential expressibility of formulas in the simplest non-trivial super-intuitionistic propositional logic

Andrei Rusu, Elena Rusu

Abstract

We consider the well-known 3-valued extension of the intuitionistic propositional logic [1] and examine the conditions for o system of formulas to be complete with respect to existential expressibility of formulas considered earlier by A. V. Kuznetsov [2]. It was established that there exists a relative simple algorithm to determine whether a system of formulas is complete relative to existential expressibility of formulas in the 3-valued extension of the intuitionistic propositional logic.

Keywords: intuitionistic logic, existential expressibility, super-intuitionistic logic.

1 Introduction

In 1921 E. Post analysed the possibility get a formula from other formulas by means of superpositions [3, 4] and proved that there are a numerable collection of closed with respect to superpositions classes of boolean functions, among which only 5 of them are maximal with respect to inclusion. A. V. Kuznetsov have generalized the notion of superposion of functions to the case of formulas and put into consideration the notion of parametric expressibility as well as existential expressibility of a formula via a system of formulas in a given logic [2] and proved there finitely many precomplete with respect to parametric expressibility classes of formulas in the general 2-valued and 3-valued logics. It was stated in [2] that together with parametric expressibility it is also interesting to investigate the existential expressibility of formulas. The main result of the present paper is the theorem that states that there is an algorithm which allows to determine whether any formula of the simplest non-trivial super-intuitionistic logic L could be existentially expressible via a given system of formulas Σ in L.

2 Definitions and notations

Intuitionistic propositional logic Int **[5].** The calculus of the propositional intuitionistic logic Int is based on formulas built as usual from propositional variables $p, q, r, p_1, q_i, r_j, \dots$, logical connectives $\&, \vee, \supset, \neg$ and auxiliary symbols of left and right parentheses (and). Axioms of Int are the formulas: $p \supset (q \supset p)$, $(p \supset q) \supset ((p \supset (q \supset r)) \supset (p \supset r)$, $p \supset (q \supset (p \,\&\, q))$, $p \supset (p \vee q)$, $p \supset (q \vee p)$, $(p \vee q) \supset p$, $(p \vee q) \supset q$, $(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$, $(p \supset q) \supset ((p \supset \neg q) \supset \neg p)$, $\neg p \supset (p \supset q)$. and the well-known rules of inference: *modus ponens*, and *substitution*. The intuitionistic logic Int of the above calculus is defined as usual as the set of formulas deductible in that calculus.

Any set of formulas L containing Int and closed with respect to the rules of inference is said to be an *extention of* Int, also being known as *super-intuitionistic logic* or *intermediate logic* [6]. We consider the super-intuitionistic logic L3 of the second slice defined by two additional axioms [6]:

$$Z = (p \supset q) \vee (q \supset p),$$

$$P_2 = ((r \supset [((q \supset p) \supset q) \supset q]) \supset r) \supset r$$

Existential expressibility [2]. Suppose in the logic L we can define the equivalence of two formulas. The formula F is said to be *(expricitly) expressible* via a system of formulas Σ in the logic L if F can be obtained from variables and formulas Σ using two rules: a) the rule of weak substitution, which allows to pass from two formulas, say A and B to the result of substitution of one of them in another in place of any variable $\frac{A,B}{A[B]}$ (where we denote by $A[B]$ the thought substitution); b) if we already get formula A and we know A is equivalent in L to B, then we have also formula B.

The formula F is said to be *existentially expressible* in the logic L via the system of formulas Σ if there exists variables $q_1, \ldots, q_s, q$ not occurring in F, formulas $D_1, \ldots, D_s$ and formulas $B_1, \ldots, B_m$ and $C_1, \ldots, C_m$ such that $B_{j1}, \ldots, B_{jm}$ and $C_{j1}, \ldots, C_{jm}$, $j = 1, \ldots, k$, are explicitly expressible in L via formulas of Σ and the following first-order formulas are true:

$$(F = q) \Longrightarrow (\bigvee_{j=1}^{k} \bigwedge_{i=1}^{m} (B_{ji} = C_{ji}))[q_1/D_1] \ldots [q_s/D_s],$$

$$(\bigvee_{j=1}^{k} \bigwedge_{i=1}^{m} (B_{ji} = C_{ji})) \Longrightarrow (F = q)$$

The system of formulas Σ is said to be *complete with respect to existential expressibility in the logic L* if any formula of the calculus of L is existentially expressible via formulas of Σ.

3 Main result

One of the main questions regarding existential expressibility of formulas is whether there is an algorithm for detecting in the given logic L able to detct the completeness with respect to existential expressibility of classes of formulas in L.

Theorem 1. *There is an algorithm for which is able to detect whether a given system formulas Σ is complete with respect to existential expressibility of formulas in the simplest three-valued extension of the intuitionistic logic.*

4 Conclusion

This is the first step in establishing the conditions for an arbitrary system of formulas Σ to be complete with respect to existential expressibility in the intuitionistic logic of propositions.

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Reversible Automata on Finite Quasigroups

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Abstract

In the given paper families of semi-automata, Mealy and Moore automata defined on finite abstract quasigroups, and on T-quasigroups are introduced and analyzed. Some applications of proposed models for resolving problems of information protection are discussed briefly.

Keywords: automata, quasigroups, T-quasigroups, hash-functions.

1 Introduction

Within the last two decades there is steady transition from combinatorial models to algebraic ones for resolving different problems of information processing, in particular, of information protection. In the majority of researches finite associative algebraic systems have been used. However, it seems very promising to use non-associative algebraic systems for resolving information protection problems. The main argument in favor of this statement is based on much more high complexity of resolving identification problems in non-associative systems in comparison with associative ones.

Among non-associative algebraic systems a specific place is held by quasigroups, which have been applied successfully for resolving model cryptography problems [1]. For this reason research of reversible automata models over finite quasigroups is important from theoretical and application-oriented point of view. In what follows we deal only with finite quasigroups, and the word "finite" is omitted.

2 Semi-Automata on Quasigroups

A semi-automaton is a triple $M = (Q, X, \delta)$, where Q is finite set of states, X is finite input alphabet, and $\delta : Q \times X \to Q$ is the transition mapping.

With any given quasigroup $\mathcal{Q} = (Q, \circ)$ the family of semi-automata $\mathfrak{M}_{\mathcal{Q}} = \{M_\diamond = (Q, Q, \diamond)\}_{\diamond \in \mathcal{P}_{\mathcal{Q}}}$, where $\mathcal{P}_{\mathcal{Q}} = \{\circ, \circ^{(r)}, \circ^{(l)}, \circ^{(rl)}, \circ^{(lr)}, \circ^{(s)}\}$ is the set of parastrophs of the quasigroup $\mathcal{Q}$, can be associated. The following theorem holds.

Theorem 1. *For any quasigroup $\mathcal{Q} = (Q, \circ)$ the transition diagram of each semi-automaton $M_\diamond \in \mathfrak{M}_{\mathcal{Q}}$ is labeled directed complete $|Q|$-graph with the loop at each vertex.*

Application-oriented meaning of this theorem is as follows.

For any semi-automaton $M_\diamond \in \mathfrak{M}_{\mathcal{Q}}$, any its state $q \in Q$, and any input string $p = x_1 \ldots x_n \in Q^n$ $(n \in \mathbb{N})$ we set

$$q \diamond p = ((\ldots((q \diamond x_1) \diamond x_2) \diamond \ldots) \diamond x_{n-1}) \diamond x_n.$$

Let $\mathrm{P}^{(1)}_{M_\diamond,q,n}(q')$ $(M_\diamond \in \mathfrak{M}_{\mathcal{Q}}; q, q' \in Q; n \in \mathbb{N})$ be probability that randomly chosen input string $p \in Q^n$ is a solution of the equation $q \diamond p = q'$, and $\mathrm{P}^{(2)}_{M_\diamond,q,n}$ $(M_\diamond \in \mathfrak{M}_{\mathcal{Q}}; q \in Q; n \in \mathbb{N})$ be probability that two different input strings p and p' randomly chosen from the set Q^n form a solution of the equation $q \diamond p = q \diamond p'$. Applying approach proposed in [2], we get the following corollary of theorem 1.

Corollary 1. *For any quasigroup $\mathcal{Q} = (Q, \circ)$ the following equalities hold*

$$\mathrm{P}^{(1)}_{M_\diamond,q,n}(q') = |Q|^{-1} \quad (M_\diamond \in \mathfrak{M}_{\mathcal{Q}}; q, q' \in Q; n \in \mathbb{N}),$$

$$\mathrm{P}^{(2)}_{M_\diamond,q,n} = |Q|^{-1} \left(1 - \frac{|Q| - 1}{|Q|^n - 1}\right) \;\; (M_\diamond \in \mathfrak{M}_{\mathcal{Q}}; q \in Q; n \in \mathbb{N}).$$

This result proves that any semi-automaton $M_\diamond \in \mathfrak{M}_{\mathcal{Q}}$ can be used as mathematical model for some family of computationally secured iterative hash-functions.

3 Automata on Quasigroups

With any given pair of quasigroups $\mathcal{Q}_i = (Q, \circ_i)$ $(i = 1, 2)$ the family of Mealy automata $\mathfrak{A}_{\mathcal{Q}_1,\mathcal{Q}_2} = \{A_{\diamond_1,\diamond_2} = (Q, Q, Q, \diamond_1, \diamond_2)\}_{\diamond_1\in\mathcal{P}_{\mathcal{Q}_1},\diamond_2\in\mathcal{P}_{\mathcal{Q}_2}}$ can be associated. The following theorem holds.

Theorem 2. *For any given pair of quasigroups $\mathcal{Q}_i = (Q, \circ_i)$ $(i = 1, 2)$ the family $\mathfrak{A}_{\mathcal{Q}_1,\mathcal{Q}_2}$ consists of reduced reversible automata.*

Corollary 2. *For any automaton $A_{\diamond_1,\diamond_2} \in \mathfrak{A}_{\mathcal{Q}_1,\mathcal{Q}_2}$ each input symbol is a distinguishing sequence.*

With any given quasigroup $\mathcal{Q} = (Q, \circ)$ and symmetric group $\mathcal{S}_Q$ the family of Moore automata $\mathfrak{A}_{\mathcal{Q},\mathcal{S}_Q} = \{A_{\diamond,f} = (Q, Q, Q, \diamond, f)\}_{\diamond\in\mathcal{P}_{\mathcal{Q}},f\in\mathcal{S}_Q}$ can be associated. The following theorem holds.

Theorem 3. *For any given quasigroup $\mathcal{Q} = (Q, \circ)$ and symmetric group $\mathcal{S}_Q$ the family $\mathfrak{A}_{\mathcal{Q},\mathcal{S}_Q}$ consists of reduced reversible automata.*

Corollary 3. *For any automaton $A_{\diamond,f} \in \mathfrak{A}_{\mathcal{Q},\mathcal{S}_Q}$ each input symbol is a distinguishing sequence.*

Application-oriented meaning of theorems 2 and 3 is that any automaton $A \in \mathfrak{A}_{\mathcal{Q}_1,\mathcal{Q}_2} \cup \mathfrak{A}_{\mathcal{Q},\mathcal{S}_Q}$ can be used as mathematical model for some stream cipher.

4 Automata on T-Quasigroups

A quasigroup $\mathcal{Q} = (Q, \circ)$ is a T-quasigroup if there exist some abelian group $\mathcal{G} = (Q, +)$, an ordered pair of its automorphisms (φ, ψ), and some fixed element $c \in Q$, such that identity $a \circ b = \varphi(a) + \psi(b) + c$ holds for all $a, b \in Q$. Thus, any T-quasigroup $\mathcal{Q} = (Q, \circ)$ can be presented as the system $\mathfrak{S} = (Q, +, \varphi, \psi, c)$.

Applying fundamental theorem of finite abelian groups, we can present abelian group $\mathcal{G} = (Q, +)$ $(|Q| = p_1^{r_1} \ldots p_t^{r_t})$ as the direct sum of cyclic subgroups of prime-power order $\mathcal{G} \cong \bigoplus_{i=1}^{t} \left(\bigoplus_{j=1}^{k_j} \mathbb{Z}_{p_j^{c_{ij}}} \right)$, where

$1 \leq c_{i1} \leq \cdots \leq c_{ik_i}$, and $r_i = c_{i1} + \cdots + c_{ik_i}$ for all $i = 1, \ldots, t$. This presentation forms the strong base for design structured models of semi-automata and reversible automata defined on finite T-quasigroups.

5 Conclusion

It has been illustrated that proposed families of semi-automata, Mealy and Moore automata, defined on finite quasigroups, can be used as mathematical models for resolving problems of information protection. Thus, it is actual detailed analysis of computational security and reliability of proposed models for different specific finite T-quasigroups. This is the main trend for future research.

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On a generalization of the inner mapping group

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Abstract

We consider the group $GM(Q,\cdot)$, generated by all left, right and middle translations of a loop $(Q,\cdot)$. The generalized inner mapping group J consists of all mappings $\alpha \in GM(Q,\cdot)$, such that $\alpha(e) = e$, where e is the unit of $(Q,\cdot)$. In the present note we give a set of mappings which generates the group J .

Keywords: loop, generalized multiplication group, inner mapping, inner mapping group.

1 Introduction

Let $(Q,\cdot)$ be a quasigroup, $h \in Q$ and let $M(Q,\cdot) =< R_x, L_y | x, y \in Q >$ be the multiplication group of $(Q,\cdot)$, where $R_x(u) = u \cdot x, L_y(u) = y \cdot u, \forall u \in Q$. A mapping $\alpha \in M(Q,\cdot)$ is called an inner mapping, with respect to h, if $\alpha(h) = h$. The group I_h of all inner mappings, with respect to h, is called the inner mapping group of $(Q,\cdot)$, with respect to h. If $h_1, h_2 \in Q$, then the inner mapping groups I_{h_1} and I_{h_2} are isomorphic. It is well known the role of I_h in the study of normality: a subquasigroup H of a quasigroup $(Q,\cdot)$ is normal if and only if $I_h(H) = H$, where h is an arbitrary element of H. A set of generators for the inner mapping group I_h of a quasigroup was given by Belousov [1].

If $(Q,\cdot)$ is a loop and $h = e$ is its unit, then I_e is called the inner mapping group of $(Q,\cdot)$. The multiplication group and the inner mapping group of a loop are important tools when studying the properties and the structure of the loops. These tools were introduced by Bruck, who used them to investigate centrally nilpotent loops. The inner mapping group of loops is studied in a wide series of works, by Bruck, Baer, Garrison, Medoch, Niemenmaa, Kepka, Nagy, Drapal, Vojtechovsky, Csorgo, Gagola and others (see, for example, [1-3]).

In the present note we consider the group $GM(Q,\cdot)$, generated by all left, right and middle translations of a loop $(Q,\cdot)$, and give a set of mappings which generates the stabilizer of the neutral element of this loop in $GM(Q,\cdot)$.

2 The generalized multiplication group of a loop and the inner mapping group

A set of generators of the inner mapping group of a quasigroup (with respect to some element) was given by Belousov [1].

Theorem 1.[1] *If $(Q,\cdot)$ is a quasigroup and $h \in Q$, than the set $\{R_{x,y}, L_{x,y}, T_x | x,y \in Q\}$ generates the inner mapping group I_h, where $R_{x,y} = R^{-1}_{x*y}R_yR_x$, $L_{x,y} = L^{-1}_{x\circ y}L_xL_y$, $T_x = L^{-1}_{\sigma(x)}R_x$, $(*) = (\cdot)^{(L_h,\varepsilon,L_h)}$, $(\circ) = (\cdot)^{(\varepsilon,R_h,R_h)}$, $\sigma = R_h^{-1}L_h$.*

For a loop $(Q,\cdot)$, we consider the group:

$$GM(Q,\cdot) =< L_x, R_y, I_z | x,y,z \in Q >,$$

where $R_x(u) = u \cdot x$, $L_y(u) = y \cdot u$, $I_z : Q \mapsto Q, I_z(u) = u \setminus z$, $\forall u \in Q$, are the right, left and, respectively, middle translations of $(Q,\cdot)$.

Let e is the the unit of a loop $(Q,\cdot)$. Denote the stabilizer of e in $GM(Q,\cdot)$ by

$$J = \{\alpha \in GM(Q,\cdot) | \alpha(e) = e\}.$$

The group J is a generalization of the inner mapping group of $(Q,\cdot)$.

Theorem 2.[4] *If $(Q,\cdot)$ and $(Q,\circ)$ are two isostrophic loops then $GM(Q,\cdot) \cong GM(Q,\circ)$*

Let $(Q,\cdot)$ be a loop. Consider the mappings: $R_{x,y} = R^{-1}_{x\cdot y}R_yR_x$, $L_{x,y} = L^{-1}_{x\cdot y}L_xL_y$,$T_x = L^{-1}_xR_x$,$P_{x,y} = R^{-1}_yL_xI_yI_x$,$P'_{x,y} = L^{-1}_xR_yI^{-1}_xI^{-1}_y$, $V_x = I^{-1}_xR_x$, $U_x = I_xR_x$, for every $x,y \in Q$.

Lemma 1. *If $(Q,\cdot)$ is a loop, then the mappings $R_{x,y}, L_{x,y}$, T_x, $P_{x,y}, P'_{x,y}$, V_x, U_x belong to its generalized inner mapping group J.*

Proof. It is well known (and easy to verify) that $R_{x,y}(e) = e, L_{x,y}(e) = e$, and $T_x(e) = e$, where e is the unit of the loop $(Q, \cdot)$. For the remaining four mappings we have:

$V_x(e) = I_x^{-1}R_x(e) = I_x^{-1}(x) = x/x = e$; $U_x(e) = I_xR_x(e) = I_x(x) = x \setminus x = e, \forall x \in Q$;

$x = e\backslash x \Leftrightarrow x \setminus y = (e\backslash x) \setminus y \Leftrightarrow x \cdot [(e\backslash x) \setminus y] = y \Leftrightarrow L_xI_yI_x(e) = R_y(e) \Leftrightarrow R_y^{-1}L_xI_yI_x(e) = e \Leftrightarrow P_{x,y}(e) = e, \forall x, y \in Q$;

$y = y/e \Leftrightarrow x/y = x/(y/e) \Leftrightarrow [x/(y/e)] \cdot y = x \Leftrightarrow R_yI_x^{-1}I_y^{-1}(e) = L_x(e) \Leftrightarrow L_x^{-1}R_yI_x^{-1}I_y^{-1}(e) = e \Leftrightarrow P'_{x,y}(e) = e, \forall x, y \in Q$.

Theorem 3. *The generalized inner mapping group J of a loop $(Q, \cdot)$ is generated by the set of mappings $\{R_{x,y}, L_{x,y}, T_x, P_{x,y}, P'_{x,y}, V_x, U_x | x, y \in Q\}$.* *Proof.* Denote $F =< R_{x,y}, L_{x,y}, T_x, P_{x,y}, P'_{x,y}, V_x, U_x | x, y \in Q >$. According to Lemma 1, all generators of F belong to J, hence $F \subseteq J$.

Every mapping $\alpha \in J$ is a products of a finite number of translations (left, right, middle) or their inverses: $S_{a_1}^{\varepsilon_1}S_{a_2}^{\varepsilon_2}\dots S_{a_n}^{\varepsilon_n}$, where $a_i \in Q$, and $\varepsilon_i = 1$ or -1, $\forall i = 1, 2, \dots, n$. The number n is called the length of α. We may consider that $S_{a_n}^{\varepsilon_n} = R_x$, for some $x \in Q$, otherwise we may add the product $R_x^{-1}R_x$ to the right side of α. The proof of the inclusion $J \subseteq F$ is similar to that of Belousov for Theorem 1, and uses the mathematical induction by the length n of α.

If $n = 1$, then $\alpha = R_x$. So as $R_x(e) = e$, we get $x = e$ and, for example, $\alpha = L_e^{-1}R_e \in F$. Suppose that every mapping $\alpha \in J$, of length $\leq n$, belongs to F.

Let $\alpha \in J$ be a mapping of length n. We'll show that α may be represented in the form $\alpha = \alpha'\tau$, where α' is a mapping of length $n-1$ and $\tau \in F$. Then, using the mathematical induction, we have $\alpha' \in F$, hence $\alpha = \alpha'\tau \in F$. So as the last factor in the product equal to α is R_x, we have to consider six possible cases: 1. $\alpha = \alpha''R_yR_x$; 2. $\alpha = \alpha''L_yR_x$; 3. $\alpha = \alpha''R_y^{-1}R_x$; 4. $\alpha = \alpha''L_y^{-1}R_x$; 5. $\alpha = \alpha''I_yR_x$; 6. $\alpha = \alpha''I_y^{-1}R_x$, where $\alpha'' \in J$ is a mapping of length $n-2$.

1. $\alpha = \alpha''R_yR_x = \alpha''R_{x\cdot y}R_{x\cdot y}^{-1}R_yR_x = \alpha''R_{x\cdot y}R_{x,y} \in F$, so as $\alpha''R_{x\cdot y}$ is a mapping of length $n-1$ (we use the inductive assumption) and $R_{x,y} \in F$.

2. $\alpha = \alpha''L_yR_x = \alpha''L_yL_xL_x^{-1}R_x = \alpha''L_yL_xT_x = \alpha''L_{x\cdot y}L_{x\cdot y}^{-1}L_yL_xT_x$

$= \alpha'' L_{x\cdot y} L_{x,y} T_x \in F$, so as $\alpha'' L_{x\cdot y}$ is of length $n-1$ and $L_{x,y}T_x \in F$.

3. $\alpha = \alpha'' R_y^{-1} R_x = \alpha'' R_z R_z^{-1} R_y^{-1} R_x = \alpha'' R_z (R_x^{-1} R_y R_z)^{-1} = \alpha'' R_z R_{z,y}^{-1} \in F$, by the inductive assumption, where $x = z \cdot y$.

4. $\alpha = \alpha'' L_y^{-1} R_x = \alpha'' L_y^{-1} L_x L_x^{-1} R_x = \alpha'' L_y^{-1} L_x T_x = \alpha'' L_z L_z^{-1} L_y^{-1} L_x T_x = \alpha'' L_z (L_x^{-1} L_y L_z)^{-1} T_x = \alpha'' L_z (L_{y,z})^{-1} T_x \in F$, where $x = t \cdot z$.

5. $\alpha = \alpha'' I_y R_x = \alpha'' I_y I_x I_x^{-1} R_x = \alpha'' I_y I_x V_x = \alpha'' L_x^{-1} R_y P_{x,y} V_x = \alpha'' L_x^{-1} L_y L_y^{-1} R_y P_{x,y} V_x = \alpha'' L_x^{-1} L_y T_y P_{x,y} V_x = \alpha'' L_z L_z^{-1} L_x^{-1} L_y T_y P_{x,y} V_x = \alpha'' L_z (L_y^{-1} L_x L_z)^{-1} T_y P_{x,y} V_x = \alpha'' L_z L_{x,z}^{-1} T_y P_{x,y} V_x \in F$, where $y = x \cdot z$.

6. $\alpha = \alpha'' I_y^{-1} R_x = \alpha'' I_y^{-1} I_x^{-1} I_x R_x = \alpha'' I_y^{-1} I_x^{-1} U_x = \alpha'' R_x^{-1} L_y P'_{x,y} U_x$

$$= \alpha'' R_x^{-1} R_y R_y^{-1} L_y P'_{x,y} U_x = \alpha'' R_x^{-1} R_y T_y^{-1} P'_{x,y} U_x =$$
$$\alpha'' R_z R_z^{-1} R_x^{-1} R_y T_y^{-1} P'_{x,y} U_x = \alpha'' R_z (R_y^{-1} R_x R_z)^{-1} T_y^{-1} P'_{x,y} U_x =$$
$$\alpha'' R_z (R_{z,x})^{-1} T_y^{-1} P'_{x,y} U_x \in F,$$

where $y = z \cdot x$, so $J \subseteq F$, which implies $J = F$.

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About one special inversion matrix of non-symmetric n-IP-loop

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Abstract

It is known that n-IP-quasigroups have more than one inversion matrix [1]. It is proved that one of these inversion matrices in the class of non-symmetric n-IP-loops is so-called matrix $[I_{ij}]$ of permutations, any of which has order two and fixes the unit element of the loop.

Keywords: quasigroup, loop, n-IP-quasigroup, n-IP-loop, inversion permutation, inversion matrix, isostrophism.

1 Main concepts and definitions

A quasigroup $Q(A)$ of arity n, $n \geq 2$, is called an n-IP-quasigroup if there exist permutations ν_{ij}, $i, j \in \overline{1,n}$ of the set Q, such that the following identities are true:

$$A(\{\nu_{ij}x_j\}_{j=1}^{i-1}, A(x_1^n), \{\nu_{ij}x_j\}_{j=i+1}^{n}) = x_i, \tag{1}$$

for all $x_1^n \in Q^n$, where $\nu_{ii} = \nu_{i\,n+1} = \varepsilon$. Here ε denotes the identity permutation of the set Q [1]. See [1] for more information on n-ary quasigroups.

The matrix

$$\left[\nu_{ij}\right] = \begin{bmatrix} \varepsilon & \nu_{12} & \nu_{13} & \dots & \nu_{1n} & \varepsilon \\ \nu_{21} & \varepsilon & \nu_{23} & \dots & \nu_{2n} & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \nu_{n1} & \nu_{n2} & \nu_{n3} & \dots & \varepsilon & \varepsilon \end{bmatrix}$$

is called *an inversion matrix* for a n-IP-quasigroup, the permutations $\nu_{i,j}$ are called *inversion permutations.* Any i-th row of an inversion matrix is called *i-th inversion system* for a n-IP-quasigroup.

The least common multiple (LCM) of orders of permutations of i-th inversion system is called the order of this system. The least common multiple (LCM) of orders of all inversion systems is called the order of inversion matrix.

The operation

$$B(x_1^n) = \alpha_{n+1}^{-1} A(\alpha_1 x_1, \dots, \alpha_n x_n),$$

for all $x_1^n \in Q$, where α_1^{n+1} are permutations of the set Q, is called an isotope of the n-ary quasigroup $Q(A)$. If $A = B$, then we have an autotopy of the n-ary quasigroup $Q(A)$.

Recall that an n-ary quasigroup is an n-ary groupoid $Q(A)$, such that in the equality $A(x_1, x_2, \dots, x_n) = x_{n+1}$ any n elements of the set $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ uniquely specifies the remaining one [1]. Therefore we can define a new quasigroup operation

$$^{\pi_i}A(x_1^{i-1}, x_{n+1}, x_{i+1}^n) = x_i, \tag{2}$$

that is called *the i-th inverse operation of the operation A.*

Let σ be a permutation of a set that consists from $(n+1)$ elements. The operation

$$^{\sigma}A(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)}$$

is called the σ-parastrophe of the operation A. If $\sigma(n+1) = n+1$, then we call this parastrophe a main parastrophe.

Isostrophy is a combination of an isotopy T and a parastrophy σ, i.e., an isostrophic image of an n-ary quasigroup $Q(A)$ is a parastrophic image of its isotopic image, and it is denoted by $A^{(\sigma,T)}$. If $A^{(\sigma,T)} = A$, then the pair (σ, T) is called an autostrophy of the n-ary quasigroup $Q(A)$ [1].

From identity (1) it follows that, for n-ary-IP-quasigroup $Q(A)$, the expression $T_i^2 = (\varepsilon, \nu_{i2}^2, \nu_{i3}^2, \dots, \nu_{ii-1}^2, \varepsilon, \nu_{ii+1}^2, \nu_{in}^2, \varepsilon)$ is an autotopy of the n-ary quasigroup $Q(A)$.

Therefore

$$^{\pi_i}A = A^{T_i} \tag{3}$$

and

$$A^{(\pi_i,T_i)} = A \tag{4}$$

for all $i \in \overline{1,n}$. Any of equalities (3) and (4) defines an n-IP-quasigroup.

Below, for convenience, we denote the operation A by $()$.

An element e is called a unit of the n-ary operation $Q()$, if the following equality is true: $(\overset{i-1}{e}, x, \overset{n-i}{e}) = x$, for all $x \in Q$ and $i \in \overline{1,n}$. n-Ary quasigroups with unit elements are called n-ary loops [1, 2]. Loops of arity $n > 2$ can have more than one unit element [1]. n-IP-quasigroups with an least one unit element are called n-IP-loops [2, 3].

Permutations I_{ij} of the set Q are defined by the equalities

$$(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}) = e,$$

for all $x \in Q$ and $i, j \in \overline{1,n}$.

If the tuple $(\varepsilon, \nu_{12}, \nu_{13}, \dots, \nu_{1n}, \varepsilon)$ is the first inversion system of n-IP-quasigroup $Q()$, with the inversion matrix $[\nu_{ij}]$, then the tuple

$$(\varepsilon, \nu_{12}^{2n-1}, \nu_{13}^{2n-1}, \dots, \nu_{1n}^{2n-1}, \varepsilon),$$

is also an (first) inversion system, since the tuple $(\varepsilon, \nu_{12}^{2n}, \nu_{13}^{2n}, \dots, \nu_{1n}^{2n}, \varepsilon)$ is an autotopy of the quasigroup $Q()$. This is true for other ($i = 2, 3, \dots$) inversion systems.

Consider the matrix

$$[I_{ij}] = \begin{bmatrix} \varepsilon & I_{12} & I_{13} & \dots & I_{1n} & \varepsilon \\ I_{21} & \varepsilon & I_{23} & \dots & I_{2n} & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ I_{n1} & I_{n2} & I_{n3} & \dots & \varepsilon & \varepsilon \end{bmatrix}$$

An n-Quasigroup $Q(A)$ is called symmetric, if $A(x_{\varphi 1}^{\varphi n}) = A(x_1^n)$, for all $\varphi \in S_n$, where S_n is the symmetric group defined on the set Q, otherwise it is called non-symmetric [2, 3].

2 Main results

The first constructed example of an 3-IP-loop have the inversion matrix $[I_{ij}]$. V.D. Belousov proposed the following problem: is it true that any n-IP-loop has among inversion matrices the matrix $[I_{ij}]$?

Lemma. *If $Q()$ is a non-symmetric n-IP-loop with the inversion matrix $[\nu_{ij}]$ and unit e, then any non-identity inversion permutation from any inversion matrix of even order does not fix the unit element e.*

Corollary. *If $Q()$ is a non-symmetric n-IP-loop with the inversion matrix $[\nu_{ij}]$ and unit e, then any non-identity inversion permutation from any inversion matrix of odd order fix the unit element e.*

Theorem. *The matrix $[I_{ij}]$ is one of the inversion matrices in any non-symmetric n-IP-loop.*

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On the lattice of ideals of semirings of continuous partial real-valued functions

E. M. Vechtomov, E. N. Lubyagina

Abstract

The work is devoted to the general theory of semirings of continuous functions. We consider semirings $CP(X)$ of continuous partial functions on topological spaces X with values in the topological field $\mathbf{R}$ of real numbers. We study properties of the lattice $\mathrm{Id}CP(X)$ of all ideals of a semiring $CP(X)$. It is proved that a T_1-space X is determined by the lattice $\mathrm{Id}CP(X)$.

Keywords: semiring, ideal, lattice, T_1-space, field of real numbers, continuous partial function, definability.

The theory of semirings of continuous functions [8–10] is a logical extension and development of the classical theory of rings of continuous real-valued functions [1]. Semirings of continuous partial functions are described in works [3–7].

Let S be a semiring and X be an arbitrary set. By SP^X we denote the set $\cup\{S^Y : Y \subseteq X\}$ of all partial functions from X to S. $D(f)$ is the domain of a partial function $f \in SP^X$.

The set SP^X with pointwise operations of addition $+$ and multiplication $\cdot$ of functions such that $D(f+g) = D(f \cdot g) = D(f) \cap D(g)$ is a semiring with an absorbing element $\varnothing$ for addition and multiplication.

Let S be a semiring with a unit 1. The idempotents (under multiplication) 1_A, $A \subseteq X$: $D(1_A) = A$, $1_A(x) = 1$ for all $x \in A$, play an important role in the theory of semirings SP^X. In particular, we have idempotents $1_x = 1_{\{x\}}$ for $x \in X$, $1_X = 1$ and $1_\varnothing = \varnothing$. The element 1_A is a unit of semirings S^A and SP^A. The semiring $SP^A = 1_A SP^X$ is a principal ideal of a semiring SP^X.

Let S contains 0. By 0_A, where $A \subseteq X$, we denote a function from SP^X such that $D(0_A) = A$ and $0_A(x) = 0$ for all $x \in X$. We have $1_\varnothing = \varnothing = 0_\varnothing$.

Let S be a topological semiring and X be a topological space. We get the semiring $CP(X,S) = \cup\{C(Y,S)\colon Y \subseteq X\}$ of all continuous partial S-valued functions on X with pointwise operations of addition and multiplication defined for any partial functions f and g on their common domain $D(f) \cap D(g)$.

Let us consider some aspects of the theory of semirings $CP(X) = CP(X, \mathbf{R})$ of continuous partial $\mathbf{R}$-valued functions on T_1–spaces X.

The set $IdCP(X)$ of all ideals of a semiring $CP(X)$ with respect to the inclusion relation $\subseteq$ forms a lattice in which the greatest lower bound of ideals is their intersection, and $\sup(A,B) = A \cup B \cup (A+B)$. The lattice $\mathrm{Id}CP(X)$ is complete one with the smallest element $\{\varnothing\}$ and the greatest element $CP(X)$.

Proposition 1. *The minimal ideals of a semiring $CP(X)$ are exactly principal ideals $(0_x) = 0_x CP(X) = \{0_x, \varnothing\}$ for all points $x \in X$.*

Proposition 2. *For any topological space X maximal ideals of a semiring $CP(X)$ have the form $(CP(X) \setminus C(X)) \cup M$, where M is an arbitrary maximal ideal of the semiring $C(X)$.*

Proposition 3. *For any topological space X the lattice $\mathrm{Id}CP(X)$ is modular.*

Proposition 4. *For any topological space X the lattice $\mathrm{Id}CP(X)$ is a lattice with pseudocomplements. Only elements $\{\varnothing\}$ and $CP(X,S)$ have a complement in $\mathrm{Id}CP(X)$.*

A space X is called an F-*space* if any two disjoint cozero-sets $\mathrm{coz} f = X \setminus \mathrm{Z}(f)$ and $\mathrm{coz} g$, $f, g \in C(X)$, are separated by a fuction from $C(X)$. A topological space X is called a *hereditary* F-*space* if any subspace of X is an F-space

Theorem 1. *For any topological space X the lattice of all ideals $\mathrm{Id}CP(X)$ is distributive if and only if X is a hereditary* F-*space.*

The space X is called a T_1-*space* if all its one-element subsets are closed. The following result is related to the topic of definability of topological spaces [2].

Theorem 2. *Any T_1-space X is uniquely determined by the lattice* $\mathrm{Id}CP(X)$.

Further we plan to develope the theory of semirings $CP(X, S)$ for different topological semirings S, in particular for the semifield (with zero 0) of nonnegative real numbers and the semifield of positive real numbers.

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Pairing on Lorentz formal modules

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Abstract

We show the main properties of this pairing — invariance and bilinearity that allow to construct the explicit form of generalized Hilbert symbol for the formal Lorentz groups over the rings of integers of local field using this pairing.

Keywords: formal modules, Hilbert pairings, explicit formulas, local fields.

1 Introduction

One of the classical examples of formal group laws is so called Lorentz formal group law:

$$F_{l,c}(X,Y) = \frac{X+Y}{1+c^2XY}.$$

Formal laws of a such type are correspondent to the relativity theory: putting here $\frac{1}{c}$ as velocity of light we get the parallel velocity addition formula.

Also if one consider curves of type $Y^2Z = X^3 + c^2X^2Z$ in Weierstrass' parametrisation on projective plane then formal Lorentz groups $F_{l,c}, \mathcal{O}$ with $\mathcal{O} = \mathbb{Z}[c]$ will correspond standard geometric point addition structure on these (algebraic) curves (see [1]).

Next we construct explicit pairing in Cartier series for the formal Lorentz group $(X+Y+XY)/(1+cXY)$ where c is some variety. Such pairings (Hilbert pairings) are important for construction of formal modules correspondent to Lorenz modules built over maximal ideal of ring of integers of local fields (finite extension of p-adic numbers). Examples of these pairings construction may be found in [2], [3].

2 Notation

The following notation is used in this paper:

- Let k be a local field, finite extension of $\mathbb{Q}_p$, p is odd prime;
- K — finite extension of k with rings of integers O_K, containing ζ — primitive root of identity of degree p^n;
- R — multiplicative system of Teichmüller representatives of residue field $\overline{K}$ of field K;
- M — maximal ideal of O_K;
- T — inertia field K/Q_p, O_T — its ring of integers;
- c, X — variables;
- $O'_T = O_T[c]$, $M_c = XO'_T[[X]]$ — ideal in $O'_T[X]$.
- Let φ be a Frobenius automorphism of $T\backslash Q_p$.
- Define Frobenius operator Δ in $O'_T[X]$:

$$\Delta(\sum a_i c^i) := \sum a_i^{\varphi} c^{pi},\ a_i \in O_T,$$

$$\Delta(\sum a_i X^i) := \sum a_i^{\Delta} X^{pi},\ a_i \in O'_T[c],$$

- $E_c,\ l_c$ — Artin – Hasse function and Vostokov function.

3 Explicit formulas of pairing

Consider multiplicative group $\mathcal{H}$ of series:

$$\mathcal{H} = \{X^m \theta\varepsilon(X)\ m \in \mathbb{Z},\ \theta \in R\},$$

with $\varepsilon(X)$ — formal series on R with free term equal to 1.

For formal series $\alpha \in \mathcal{H},\ \beta \in F_c(M_c)$ constructing pairing:

$$\langle\ ,\ \rangle_c : \qquad \mathcal{H} \times F_c(M_c) \longrightarrow O'_T \quad \text{mod } (p^n, \mathcal{P})$$

$$\alpha, \beta \longrightarrow \mathrm{res}_X \Phi(\alpha, \beta) \backslash s_{l,c},$$

with

- $\Phi(\alpha, \beta) = l_c(\beta) \cdot \alpha^{-1}\mathrm{d}\alpha - l(\alpha)c^{-1}\mathrm{d}\frac{\Delta}{p}c\lambda_c(\beta)$
- $\mathrm{d} := \frac{\mathrm{d}}{\mathrm{d}X}$
- $l(\alpha) = (1 - \frac{\Delta}{p}) \log(\alpha)$
- $s_{l,c} = [p^n]_c(\underline{\xi_n})$. Here ξ_n is a root of isogeny $[p^n]_{l,c}$ consisting in K and $\underline{\xi_n}$ is a such series that $\underline{\xi}(\pi) = \xi$ for some prime $\pi \in K$.
- $\mathcal{P}(\gamma) = \gamma^{\Delta} - \gamma$ for γ consisting in O'_T.

One may demonstrate that paring given above is well-defined, bilinear and invariant. The invariance property we give as

Theorem. For series $\alpha(X) \in \mathcal{H},\ \beta(X) \in F_c(M_c),\ g(Y) \in O'_T[[Y]]$, $g(0) = 0$ we have:

$$\langle \alpha(X), \beta(X) \rangle_c = \langle \alpha(g(Y)), \beta(g(Y)) \rangle_c.$$

4 Conclusion

In this paper, we constructed the explicit formulas of Hilbert pairing of Lorentz formal modules and deduced its standard properties.

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Explicit formula for Hilbert symbol of polynomial formal group over multidimensional local field

Sergey Vostokov, Vladislav Volkov

Abstract

This paper continues series of work on explicit formulas for Hilbert symbol of formal groups. We consider simplest possible situation in which coefficients of the group do not lie in the inertia subfield of the field in question. In other words coefficient may contain ramifications. One-dimensional situation was considered in [1].

Previous results for the multidimensional case include classical multiplicative group case described in the works [2], [3], case of Lubin-Tate groups in [4] and case of Honda groups in [5], [6].

Keywords: Hilbert symbol, multidimensional local field, formal groups.

1 Introduction

Construction of explicit formulas for Hilbert symbol is motivated by their applications in cryptography; because they provide constructive approach to the class field theory; and due to the fact that residue-type formulas are proper analogies to the residues of meromorphic function in the sense that Hilbert reciprocity law is analogue of Cauchy residue theorem.

We consider case of multidimensional local field K with characteristic different from characteristic of its residue field. This limitation is of purely technical nature, other cases can be achieved in a similar fashion.

Let us introduce following notation:
$p \geqslant 3$ — prime number;
ζ — p^mth primitive root of unity;
K — n-dimensional local field, which contains ζ; characteristic of K is required to be zero, while it's first residue field should have characteristic p. We denote its ring of integers $\mathcal{O}_K$.
c — unit element of the field K;
$t_1, \dots t_{n-1}, \pi$ — system of local parameters of the field K.
T — inertia subfield of K. Detailed definition follows below. We denote its ring of integers $\mathcal{O}_T$;
Let's also denote $\mathcal{O} = \mathcal{O}_T\{\{t_1\}\}\{\{t_2\}\}\dots\{\{t_{n-1}\}\}$.
$\triangle$ — Frobenius automorphism in $T/\mathbb{Q}_p$;
tr — trace operator in $T/\mathbb{Q}_p$;
$\mathfrak{M}$ — maximal ideal in $\mathcal{O}_K$;
$\mathfrak{R}$ — Teichmuller representatives of $K^{(0)}$ in $\mathcal{O}_T$;

Every field K with similar characteristic constraints is in fact finite extension of the field of formal series $k\{\{t_1\}\}\{\{t_2\}\}\dots\{\{t_{n-1}\}\}$, where k is some finite extension of $\mathbb{Q}_p$. Thus we can choose first $n-1$ local parameters to be independent variables.

In one-dimensional case inertia subfield T of the field K is simply maximal non-ramified subfield in $K/\mathbb{Q}_p$. In general case T is chosen as field of fractions of Witt vectors of the last residue field of K.

Consider polynomial formal group $F_c = X + Y + cXY$. It defines formal $\mathbb{Z}_p$-module $F_c(\mathfrak{M})$ over the ideal $\mathfrak{M}$ by formal addition. We denote by $[a]_c(\cdot)$ multiplication by $a \in \mathbb{Z}_p$ in this module. The kernel of the function $[p^m]_c(\cdot)\colon F_c(\mathfrak{M}) \to F_c(\mathfrak{M})$ is spawned by $\xi = c^{-1}(\zeta - 1)$.

Now we consider Parshin-Kato isomorphism between K-Milnor group of multiplicative group K^* and Galois group of maximal Abelian extension of K:

$$\Xi\colon K_n(K^*) \to \mathrm{Gal}(K^{ab}/K)\,,$$

where K_n is nth Milnor group. In one-dimensional case Ξ is simply an Artin map for the local class field theory. By abuse of notation let's denote through $[p^m]_c^{-1}(\beta)$ any of the solutions of the equation $[p^m]_c(t) = \beta$, where $\beta \in K$. Then by analogy to classical Hilbert symbol one can consider the following modification:

$$(\alpha, \beta)_c = [p^m]_c^{-1}(\beta)^{\Xi(\alpha)} -_{F_c} [p^m]_c^{-1}(\beta)\,,$$

where $\alpha \in K_n(K^*)$, $\beta \in F_c(\mathfrak{M})$. It's easy to see that the right side is well defined and actually belongs to $\langle\xi\rangle_c$, i. e. kernel of the $[p^m]_c(\cdot)$ map.

Our goal is to give constructive approach to $(\alpha, \beta)_c$, by expressing it through explicit formula. Basic plan is as follows:

- Determine convenient basis with respect to formal action.
- Explicitly construct a formal pairing analogous to Hilbert symbol on the series of formal variables (by lifting $\pi \to t_n$).
- Explicitly check that most important properties of the Hilbert symbol hold for the newly defined pairing.
- Project constructed pairing on the numbers (through $t_n \to \pi$ map) and check that it coincides with classical Hilbert pairing by comparing on the basis.

For the sake of brevity we skip first step in current description.

2 Formal series

Let's denote through $\mathcal{O}_T\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}[[t_n]]_1$ the set of all series with lexicographic degree not less than $(1, 0, 0, \ldots, 0)$ (latter degrees are more lexicographically important). We consider following formal analogs of K^* and $F_c(\mathfrak{M})$: group $\mathcal{H}_\mathfrak{m} = \langle t_1\rangle \times \ldots \times \langle t_n\rangle \times \mathfrak{R} \times (1 + \mathcal{O}_T\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}[[t_n]]_1)$ and formal module $\mathcal{H}_c = \mathcal{O}_T\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}[[t_n]]_1$.

We also extend operator $\triangle$ over all formal series from $\mathcal{O}_T\{\{t_1\}\}$ $ldots\{\{t_{n-1}\}\}((t_n))$ by stating $\triangle\,(t_i) = t_i^p$.

Definition 1. *Pairing $[\cdot,\cdot]_c$ between group ${\mathcal{H}_{\mathfrak{m}}}^n$ and module $\mathcal{H}_{\mathrm{c}}$ is defined as*

$$\begin{aligned}[\cdot,\cdot]_c\colon (\mathcal{H}_{\mathfrak{m}})^n \times \mathcal{H}_{\mathrm{c}} &\to \mathcal{O}_T/p^m\\ \alpha,\beta &\mapsto \operatorname{res}\Phi(\alpha,\beta)\,/s_c \mod p^m,\end{aligned}$$

$$\textit{where } \Phi(\alpha,\beta) = \ell_c(\beta)D'_{n+1} - \sum_{i=1}^{n} \frac{1}{p^{n-i}}\ell(\alpha_i)D'_i$$

D'_i is a specific determinant.

3 Main properties

The main properties of the pairing $[\cdot,\cdot]_c$ that we check are:

- linearity

$$\begin{aligned}[\{\ldots,\alpha_1\cdot\alpha_2,\ldots\},\beta]_c &= [\{\ldots,\alpha_1,\ldots\},\beta]_c + [\{\ldots,\alpha_2,\ldots\},\beta]_c\\ [\alpha,\beta_1 +_{F_c} \beta_2]_c &= [\alpha,\beta_1]_c + [\alpha,\beta_2]_c\\ [\alpha,[r]_c\beta]_c &= r\cdot[\alpha,\beta]_c;\end{aligned}$$

- co-symmetry

$$[\{\ldots,\alpha_1,\ldots,\alpha_2,\ldots\},\beta]_c = -[\{\ldots,\alpha_2,\ldots,\alpha_1,\ldots\},\beta]_c;$$

- Steinberg property

$$[\{\ldots,\alpha,\ldots,1-\alpha,\ldots\},\beta]_c = 0;$$

- Steinberg property

$$[\{\ldots,\alpha,\ldots\},c^{p^m-1}\alpha]_c = 0;$$

- independence of the second argument modulo $t_n \to \pi$ projection;
- independence of coordinate change.

Most of these properties can be checked by induction on dimension.

4 Projection

Due to above properties of the pairing $[\cdot,\cdot]_c$ we can induce the pairing $\langle\cdot,\cdot\rangle_c$ on $K_n(\mathcal{H}_{\mathfrak{m}}) \times \mathcal{H}_c$ from it.

We now introduce pairing $\{\cdot,\cdot\}_c$ on numbers.

Definition 2. *Let's denote*

$$\{\cdot,\cdot\}\colon K_n(K) \times F_c(\mathfrak{M}) \to \langle\xi\rangle_c$$
$$\{\alpha,\beta\}_c = \Big[\mathrm{tr}\ \ \underline{\alpha},\underline{\beta}\rangle_c\Big]_c(\xi)\,.$$

Main theorem now follows from explicit check on the basis and linearity properties.

Theorem 1. *For any elements $\alpha \in K_n(K)$ and $\beta \in F_c(\mathfrak{M})$ values of pairings $\{\cdot,\cdot\}_c$ and $(\cdot,\cdot)_c$ coincide:*

$$\{\alpha,\beta\}_c = (\alpha,\beta)_c\,.$$

5 Conclusion

In this paper the explicit formula for the Hilbert pairing of polynomial formal group is constructed.

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Free commutative (n-nilpotent) strong doppelsemigroups

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Abstract

We construct a free commutative (n-nilpotent) strong doppelsemigroup and characterize the least commutative (n-nilpotent) congruence on a free strong doppelsemigroup.

Keywords: strong doppelsemigroup, free commutative strong doppelsemigroup, free n-nilpotent strong doppelsemigroup, semigroup, congruence.

1 Preliminaries

Recall that a doppelsemigroup [1 – 4] is a nonempty set D with two binary operations $\dashv$ and $\vdash$ satisfying the axioms

$$(x \dashv y) \vdash z = x \dashv (y \vdash z),$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$

$$(x \dashv y) \dashv z = x \dashv (y \dashv z),$$

$$(x \vdash y) \vdash z = x \vdash (y \vdash z)$$

for all $x, y, z \in D$. A doppelsemigroup $(D, \dashv, \vdash)$ is called strong [2] if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z)$$

for all $x, y, z \in D$.

Recall necessary definitions from [1], [3].

As usual, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}^0$ denotes $\mathbb{N}$ with zero. An element 0 of a doppelsemigroup $(D, \dashv, \vdash)$ is called zero if $x * 0 = 0 = 0 * x$ for all $x \in D$ and $* \in \{\dashv, \vdash\}$. A doppelsemigroup $(D, \dashv, \vdash)$ with zero 0 is called nilpotent if for some $n \in \mathbb{N}$ and any $x_i \in D$ with $1 \leq i \leq n+1$, and $*_j \in \{\dashv, \vdash\}$ with $1 \leq j \leq n$,

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1} = 0.$$

The least such n is called the nilpotency index of $(D, \dashv, \vdash)$. For $k \in \mathbb{N}$ a nilpotent doppelsemigroup of nilpotency index $\leq k$ is called k-nilpotent. A doppelsemigroup $(D, \dashv, \vdash)$ is called commutative if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. The class of all commutative (n-nilpotent) strong doppelsemigroups forms a subvariety of the variety of strong doppelsemigroups. A strong doppelsemigroup which is free in the variety of commutative (n-nilpotent) strong doppelsemigroups will be called a free commutative (n-nilpotent) strong doppelsemigroup. If ρ is a congruence on a doppelsemigroup $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is a commutative (n-nilpotent) doppelsemigroup, then ρ is called a commutative (n-nilpotent) congruence.

Let X be an arbitrary nonempty set and ω an arbitrary word in the alphabet X. The length of ω will be denoted by l_ω. Let further $F[X]$ be the free semigroup in the alphabet X. Define operations $\dashv$ and $\vdash$ on $\{(w, m) \in F[X] \times \mathbb{N}^0 \mid l_w > m\}$ by

$$(w_1, m_1) \dashv (w_2, m_2) = (w_1 w_2, m_1 + m_2 + 1), \tag{1.1}$$

$$(w_1, m_1) \vdash (w_2, m_2) = (w_1 w_2, m_1 + m_2). \tag{1.2}$$

The algebra obtained in this way is denoted by $\widetilde{F}[X]$. According to [2], $\widetilde{F}[X]$ is the free strong doppelsemigroup.

2 Free objects

In this section, we construct a free commutative (n-nilpotent) strong doppelsemigroup of an arbitrary rank.

Let X be an arbitrary nonempty set and $F^*[X]$ the free commutative semigroup in the alphabet X. Define operations $\dashv$ and $\vdash$ on $C = \{(w,m) \in F^*[X] \times \mathbb{N}^0 \,|\, l_w > m\}$ by (1.1) and (1.2) for all (w_1,m_1), $(w_2,m_2) \in C$. The algebra $(C,\dashv,\vdash)$ will be denoted by $\widetilde{F^*}[X]$.

Theorem 1. *$\widetilde{F^*}[X]$ is the free commutative strong doppelsemigroup.*

Fix $n \in \mathbb{N}$ and assume $C_n = \{(w,m) \in \widetilde{F}[X] \,|\, l_w \leq n\} \cup \{0\}$. Define operations $\dashv$ and $\vdash$ on C_n by

$$(w_1,m_1) \dashv (w_2,m_2) = \begin{cases} (w_1w_2, m_1+m_2+1)\,, & l_{w_1w_2} \leq n, \\ 0, & l_{w_1w_2} > n, \end{cases}$$

$$(w_1,m_1) \vdash (w_2,m_2) = \begin{cases} (w_1w_2, m_1+m_2)\,, & l_{w_1w_2} \leq n, \\ 0, & l_{w_1w_2} > n, \end{cases}$$

$$(w_1,m_1) * 0 = 0 * (w_1,m_1) = 0 * 0 = 0$$

for all $(w_1,m_1), (w_2,m_2) \in C_n \backslash \{0\}$ and $* \in \{\dashv,\vdash\}$. The algebra $(C_n,\dashv,\vdash)$ will be denoted by $FNSD_n(X)$.

Theorem 2. *$FNSD_n(X)$ is the free n-nilpotent strong doppelsemigroup.*

We also consider separately free commutative (n-nilpotent) strong doppelsemigroups of rank 1 and establish that the automorphism groups of $\widetilde{F^*}[X]$ and $FNSD_n(X)$ are isomorphic to the symmetric group on X.

3 The least congruences on a free strong doppelsemigroup

In this section, we present the least commutative (n-nilpotent) congruence on a free strong doppelsemigroup.

Let $\widetilde{F}[X]$ be the free strong doppelsemigroup. By $\star$ denote the operation on $F^*[X]$. Take $(x_1 \ldots x_k, i), (y_1 \ldots y_h, j) \in \widetilde{F}[X]$, where $x_p, y_q \in X$ for $1 \leq p \leq k$, $1 \leq q \leq h$, and define a relation η on $\widetilde{F}[X]$ by

$$(x_1 \dots x_k, i)\, \eta\, (y_1 \dots y_h, j) \quad \text{if and only if}$$

$$x_1 \star \dots \star x_k = y_1 \star \dots \star y_h, \ i = j.$$

For every $n \in \mathbb{N}$ define a relation $\wp_{(n)}$ on $\widetilde{F}\,[X]$ by

$$(w_1, m_1)\, \wp_{(n)}\, (w_2, m_2) \quad \text{if and only if}$$

$$(w_1, m_1) = (w_2, m_2) \quad \text{or} \quad l_{w_1} > n, \, l_{w_2} > n.$$

Theorem 3. Let $\widetilde{F}[X]$ be the free strong doppelsemigroup. Then
(i) η is the least commutative congruence on $\widetilde{F}\,[X]$;
(ii) $\wp_{(n)}$ is the least n-nilpotent congruence on $\widetilde{F}\,[X]$.

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Section 2

Geometry and Topology

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The global benavior of geodesics on hyperbolic manifolds

Vladimir Balcan

Abstract

This paper focuses on the problem of the global behavior of geodesics on the arbitrary hyperbolic two-manifold, or surfaces.

Keywords: behavior of geodesics, hyperbolic pants, hyperbolic surface.

In this work for the first time systematically is described the geometry of behavior of geodesics on hyperbolic manifolds. Geodesics on smooth surfaces are the straightest and locally shortest curves. My research is to better understand geodesics on a hyperbolic surface. Much less is known about the behavior of geodesics on hyperbolic surfaces. The chaotic behavior of geodesics on surface of constant negative curvature and finite volume has been known since Hadamard (1898). Emil Artin studied the global behavior of geodesics on hyperbolic surfaces by cleverly encoding geodesics using continued fractions. A major problem we are interested in is to describe of the geodesics trajectories on two-dimensional hyperbolic manifold. We want to understand the global behavior of geodesics with a given direction. In particular a) when are geodesics closed? b) when are the dense in the surface? c) quantitatively, how do they wrap around the surface? These questions admit notably precise answers, as we are going to see. Let us recall some definitions concerning hyperbolic surfaces and geodesics.

A (closed) hyperbolic surface can be defined either by a Riemannian metric of constant negative curvature or (thanks to the uniformization theorem) by a quotient of hyperbolic plane by a discrete group of isometries, isomorphic to the fundamental group of the initial surface, acting properly discontinuously on hyperbolic plane. A standard tool in the study of compact hyperbolic surfaces is the decomposition into "pairs of pants" (Y pieces). A hyperbolic surface of signature (g, n) is an oriented, connected surface of genus g with n boundary components, called boundary geodesics, which is equipped with a metric of constant negative curvature. A hyperbolic surface of genus g with k punctures and n holes and with no boundary is said to be of type (g, n, k). A geodesic in a hyperbolic manifold is a locally distance - minimizing curve, and is said to be simple if it has no transverse self-intersections (therefore it is either an embedded copy of $\mathbb{R}$ or an embedded circle) and non simple otherwise. A geodesic on surface M is said to be complete if it is not strictly contained in any other geodesic, i.e., it is either closed and smooth, or open and of infinite length in both directions. Complete geodesics coincide with those which never intersect ∂M. Note that if M is obtained from a compact surface by removing a finite number of points to form cusps then a complete open geodesic on M might tend toward infinity along a cusp. For a hyperbolic surface M some of the geodesics γ will come back to the point they start and fit in a smooth way. These are called closed geodesics.

How do geodesics on the hyperbolic surface behave or how can we determine the behavior of a given geodesic on the hyperbolic surface? We investigate in detail the global behavior of geodesics on the simplest hyperbolic surfaces: hyperbolic horn (funnel end), hyperbolic cylinder and parabolic horn (cusp, horn end), or parabolic cylinder. A hyperbolic horn is a two-dimensional manifold, obtained from the strip between the two parallel straight lines of the hyperbolic plane by matching the border lines by shifting (sliding), its axis being parallel to he border lines and beyond the strip between them. The hyperbolic horn, i.e. the factor-space H^2_+/Γ, is an (open) half of the hyperbolic cylinder. The border circumference does not belong to that half and there for the surface of the hyperbolic horn is incomplete.

On the hyperbolic horn the problem of behavior of a geodesic is solvable.

The following types of geodesic on hyperbolic horn are identified:

1) *there are no closed geodesics*; 2) there is a geodesic of *infinite length, without self-intersections points*, and any of its points divides the geodesic into two rays: one ray of finite length and another ray of infinite length; 3) there is an *infinite geodesic, without self-intersections points* and any of its points divides it into two congruent rays; 4) *there is an infinite geodesic and it has a finite number k* of double self-intersection points *and they are all divisible by* 2. The number k of self-intersection points of an examined geodesic is equal to p. One may define the *hyperbolic cylinder* as a non-compact two-dimensional manifold obtained from the strip from between the two divergent lines of the hyperbolic plane by identifying the divergent border lines by shift (sliding), its axis being a common perpendicular for the said border lines, its shift being equal to the length of such translation.

On the hyperbolic cylinder $C = H^2/\Gamma$ the geodesic's behavior problem is solvable. There are no closed geodesics on the cylinder C (both simple, different from the narrow geodesic core of cylinder and non-simple ones). If the geodesic's image intersects the straight line a, such a geodesic is a geodesic without self-intersection points, infinite inboth directions (at both ends).

Let us consider the behavior of geodesic on a parabolic cusp (parabolic cylinder). We shall call a *parabolic horn (cusp)* the two-dimensional manifold obtained from the strip from between the two parallel lines of the hyperbolic plane by identifying the border lines by horocyclic rotation determined by these lines. The parabolic cylinder is a special case (its small end is a cusp, while the "horn" end carriers the hyperbolic metric). The problem of behavior of a geodesic on a horn end (cusp) is solvable. The study of universal cover of parabolic cusp demonstrates that: a) if the arbitrary straight line c does not cross the obstructing line of the pair determining the horocyclic rotation w and identified upon that rotation, the image of the said straight line on this surface(cusp) is isometric to the usual straight line of a hyperbolic surface *(simple infinite length, without self-intersection)*; b) if the image of the geodesic c on the hyperbolic plane H^2 is a straight line intersecting the said geodesic and if it is different from the obstructing straight line, then the geodesic c is *infinite in both directions* (at both ends) and it has only a finite number k of double self-intersection points. In the particular case, both ends of the geodesic can go to the some point at infinity; c) there are no closed geodesics on the parabolic cusp, because no translation in the group $\Gamma = < w >$.

The study of geodesics on hyperbolic surfaces can be reduced to the study of curves on a hyperbolic pair of pants. Compact hyperbolic surfaces can be seen as an elementary pasting of geodesic polygons of the hyperbolic plane. Conversely, cutting such a surface along disjoint simple closed geodesics (a partition), one obtains a family of pair of pants, which in turn can be readily cut to obtain a pair of isometric right-angled hexagons. Let M be a surface and let P be a pair of pants. In this paper, we focus on getting the behavior of geodesics on P. As a direct consequence we get the behavior of geodesics on any surface M. We do this as follows. First, there is a unique way to write P as the union of two congruent right-angled hexagons. Take this decomposition. We examining different types of behaviors exhibited by geodesics on a given pair of hyperbolic pants and study infinite simple geodesic rays and complete geodesics. We also allow the degenerate case in which one or more of the lengths vanish (a generalized pair of pants).

Main results of the present work are as follows. In the work is given a constructive method for solving the problem of the behavior of geodesic on a arbitrary hyperbolic surfaces of signature (g, n, k), i.e., method allowing to answer the question about the structure on the global of examined geodesic at its indefinitely extension on both directions. Such a compressed formulated result can be disclosed as follows. For this purpose, with the help of proposed practical approach at first are studied geodesics at the simplest hyperbolic manifolds. Investigation of behavior of geodesics on the listed above surfaces, allowed finding answer of assigned task in the most general case.

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Methods of construction of Hausdorff extensions

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Abstract

In this paper we study the extensions of Hausdorff spaces generated by discrete families of open sets.

Keywords: extension, P-space, remainder

1 Introduction

Any space is considered to be a Hausdorff space. We use the terminology from [2]. For any completely regular space X denote by βX the Stone-Čech compactification of the space X.

Fix a space X. A space eX is an extension of the space X if X is a dense subspace of eX. If eX is a compact space, then eX is called a compactification of the space X. The subspace $eX \setminus X$ is called a remainder of the extension eX.

Denote by $Ext(X)$ the family of all extensions of the space X. If X is a completely regular space, then by $Ext_\rho(X)$ we the family of all completely regular extensions of the space X. Obviously, $Ext_\rho(X) \subset Ext(X)$. Let $Y, Z \in Ext(X)$ be two extensions of the space X. We consider that $Z \leq Y$ if there exists a continuous mapping $f : Y \longrightarrow Z$ such that $f(x) = x$ for each $x \in X$. If $Z \leq Y$ and $Y \leq Z$, then we say that extensions Y and Z are equivalent and there exists a unique homeomorphism $f : Y \longrightarrow Z$ of Y onto Z such that $f(x) = x$ for each $x \in X$. We identify the equivalent extensions. In this case $Ext(X)$ and Ext_ρ are partial ordered sets.

Let τ be an infinite cardinal.

Denote by $O(\tau)$ the set of all ordinal numbers of cardinality $< \tau$. We consider that τ is the first ordinal number of the cardinality τ. For any $\alpha \in O(\tau)$ we put $O(\alpha) = \{\beta \in O(\tau) : \beta < \alpha\}$. In this case $O(\tau)$ is well ordered set such that $|O(\tau)| = \tau$ and $|O(\alpha)| < \tau$ for every $\alpha \in O(\tau)$.

A point $x \in X$ is called a $P(\tau)$-point of the space X if for any non-empty family γ of open subsets of X for which $x \in \cap\gamma$ and $|\gamma| < \tau$ there exists an open subset U of X such that $x \in U \subset \cap\gamma$. If any point of X is a $P(\tau)$-point, then we say that $P(\tau)$-space.

Any point is an $\aleph_0$-point. If $\tau = \aleph_1$, then the $P(\tau)$-point is called the P-point.

2 Hausdorff extensions of discrete spaces

Let τ be an infinite cardinal. Let E be a discrete space of the cardinality $\geq \tau$.

A family η of subsets of E is called a τ-centered if the family η is non-empty, $\cap\eta = \emptyset$, $\emptyset \notin \eta$ and any subfamily $\zeta \subset \eta$, with cardinality $|\zeta| < \tau$, there exists $l \in \eta$ such that $L \subset \cap\zeta$.

Two families η and ζ of subsets of the space E are almost disjoint if there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$.

Any family of subsets is ordered by the following order: $L \preceq H$ if and only if $H \subset L$. Relatively to this oder some families of sets are well-ordered.

Proposition 2.1. *Let $k = |E| \geq \tau$ and $\Sigma\{k^m : m < \tau\} = k$. Then on E there exists a set Ω of well-ordered almost disjoint τ-centered families such that $|\Omega = k^\tau$ and $|\eta| = \tau$ for each $\eta \in \Omega$.*

Proof. We fix an element $0 \in E$. For every $\alpha \in O(\tau)$ we put $E_\alpha = E$ and $0_\alpha = 0$. Then $E^\tau = \Pi\{E_\alpha : \alpha \in)(\tau)\}$. For each $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$ we put $\phi(x) = sup\{0, \alpha : x_\alpha \neq 0_\alpha\}$. Obviously, $0 \leq \phi(x) \leq \tau$. Let $D = \{x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau : \phi(x) < \tau\}$. By construction, $|D| = \Sigma\{k^m : m < \tau\} = k$ and $|E^\tau| = k^\tau$. Since $|E| = |D|$, we can fix a one-to-one mapping $f : E \longrightarrow D$. Fix a point $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$. For any $\beta \in)(\tau)$ we put $V(x, \beta) =$

$\{y = (y_\alpha : \alpha \in O(\tau)) \in E^\tau : y_\alpha = x_\alpha$ *for every* $\alpha \leq \beta\}$ and $\eta_x = \{L(x,\beta) = f^{-1}(D \cap V(x,\beta) : \beta \in 0(\tau)\}$. Then $\Omega = \{\eta_x : x \in E^\tau\}$ is the desired set of τ-centered families.

Remark 2.2. *Let* $|E| = k \geq \tau$. *Since on* E *there exists* k *mutually disjoint subsets of cardinality* τ, *on* E *there exists a set* Φ *of well-ordered almost disjoint* τ*-centered families such that* $|\Phi| \geq k$ *and* $|\eta| = \tau$ *for each* $\eta \in \Phi$.

Fix a set Φ of almost disjoint τ-centered families of subsets of the set E. We put $e_\Phi E = E \cup \Phi$. On $e_\Phi E$ we construct two topologies.

Topology $T^s(\Phi)$. The basis of the topology $T^s(\Phi)$ is the family $\mathcal{B}^s(\Phi) = \{U_L = L \cup \{\eta \in \Phi : H \subset L \text{ for some } H \in \eta\} : L \subset E\}$.

Topology $T_m(\Phi)$. For each $x \in E$ we put $B_m(x) = \{\{x\}\}$. For every $\eta \in \Phi$ we put $B_m(\eta) = \{V_{(\eta,L)} = \{\eta\} \cup L : L \in \eta\}$. The basis of the topology $T_m(\Phi)$ is the family $\mathcal{B}_m(\Phi) = \cup\{B_m(x) : x \in e_\Phi E\}$.

Theorem 2.3. *The spaces* $(e_\Phi E, T^s(\Phi))$ *and* $(e_\Phi E, T_m(\Phi))$ *are Hausdorff zero-dimensional extensions of the discrete space* E, *and* $T^s(\Phi) \subset T_m(\Phi))$. *In particular,* $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, T_m(\Phi))$.

Proof. The inclusion $T^s(\Phi) \subset T_m(\Phi))$ follows from the constructions of the topologies $T^s(\Phi)$ and $T_m(\Phi))$. If $L \in \eta \in \Phi$, then $\eta \in clL$. Hence the set E is dense in the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. If the families $\eta, \zeta \in \Phi$ are distinct, then there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$. Then $U_L \cap U_Z = \emptyset$. If $L \subset E$ and $|L| < \tau$, then L is an open-and-closed subset of the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. Hence the topologies $T^s(\Phi)$ and $T_m(\Phi)$ are discrete on E and the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are Hausdorff extensions of the discrete space E. Since the sets U_L and $V_{(\eta,L)}$ are open-and-closed in the topologies $T^s(\Phi)$ and $T_m(\Phi))$, respectively, the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are zero-dimensional.

Theorem 2.4. *The spaces* $(e_\Phi E, T^s(\Phi))$ *and* $(e_\Phi E, T_m(\Phi))$ *are* $P(\tau)$*-spaces.*

Proof. Fix $\eta \in \Phi$. If $\zeta \subset \eta$ and $|\zeta| < \tau$, then there exists $L(\zeta) \in \eta$ such that $L(\zeta) \subset \cap\zeta$. From this fact immediately follows that $(e_\Phi E, T_m(\Phi))$ is a $P(\tau)$-space. Assume that $\{L_\mu : \mu \in M\}$ is a family of subsets of E, $|M| < \tau$, $\eta \in \Phi$ and $\eta \in \cap\{L_\mu : \mu \in M\}$. Then there exists $L \in \eta$ such that $L \subset \cap\{L_\mu : \mu \in M\}$. Thus $\eta \in U_L \in \cap\{U_{L_\mu} : \mu \in M\}$. From this fact immediately follows that $(e_\Phi E, T^s(\Phi))$ is a $P(\tau)$-space.

Corollary 2.5. *If* $T^s(\Phi) \subset \mathcal{T} \subset T_m(\Phi))$*, then* $(e_\Phi E, \mathcal{T})$ *is a Hausdorff extension of the discrete space* E*, and* $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, \mathcal{T}) \leq (e_\Phi E, T_m(\Phi))$.

Theorem 2.6. *The space* $(e_\Omega E, T^s(\Omega))$*, where* Ω *is the set of well-ordered almost disjoint* τ*-centered families from Proposition 2.1, is a zero-dimensional paracompact space with character* $\chi(e_\Omega E, T^s(\Omega)) = \tau$ *and weight* $\Sigma\{|E|^m : m < \tau\}$.

Proof. We consider that $E = D$. The family $\mathcal{B} = \{\{x\} : x \in D\} \cup \{V(x, \beta) : x \in E^\tau, \beta \in O(\tau)\}$ is a base of the topology $T^s(\Omega)$. If $U, V \in \mathcal{B}$, then or $U \subset V$, or $V \subset U$, or $U \cap V = \emptyset$. From the A. V. Arhangel'skii theorem [1] it follows that $(e_\Omega E, T^s(\Omega))$ is a zero-dimensional paracompact space.

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Conditions of Finiteness and Algebraical Properties of Topological Spaces

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Abstract

It is determined that in a Mal'cev F-space any compact subset is finite.

Keywords: Mal'cev algebra, pseudocompact space, bounded set, F-space.

1 Introduction

Any space is considered to be a T_0-space. An n-ary operation on a space X is a mapping $p : X^n \longrightarrow X$. If the mapping p is continuous, then we say that p is a continuous operation. A Mal'cev operation on a space X is a continuous mapping $m : X^3 \to X$ such that $m(x, x, z) = z$ and $m(x, y, y) = x$ for all $x, y, z \in X$. A space is called a Mal'cev space if it admits a Mal'cev operation. Any Mal'cev space is a Hausdorff space [4].

An topological universal algebra A is a homogeneous algebra if there exist two binary derivate operations p, q such that $q(x, p(x, y)) = y$, $p(x, q(x, y)) = y$ and $p(x, x) = p(y, y)$ for all $x, y \in A$. A topological quasigroup admits a structure of a topological homogeneous algebra. Any topological homogeneous algebra is a regular space.

2 On Spaces with Metrizable Open Images

A space X is called an F-space if X is completely regular and if disjoint cozero-sets of X are contained in disjoint zero-sets; A space X is called an F'-space if X is completely regular and $cl_X U \cap cl_X V = \emptyset$ for every two disjoint functionally open sets U and V of X (see [7]).

Proposition 2.1. *Let $f : X \longrightarrow Y$ be a continuous mapping of an F'-space X onto a Tychonoff space Y, Z be a subspace of X, $f(Z) = Y$ and the restriction $g = f|Z : Z \longrightarrow Y$ is an open mapping. Then Y is an F'-space.*

Proposition 2.2. *Let Y be an F'-embedded subspace of an F'-space X. Then Y is an F'-space.*

A space is weakly Lindelöf if each of its open covers admits a countable subfamily with dense union.

Proposition 2.3. *Let Y be a weakly Lindelöf subspace of a completely regular space X. Then Y is F'-embedded in X.*

Proposition 2.4. *Let Y be a weakly Lindelöf subspace of a completely regular F'-space X. Then Y is an F-space.*

Proof. The following fact was obtained by many authors (see [5], Corollary 1.6): Any weakly Lindelöf subspace of an F'-space is itself an F'-space. That fact follows from Propositions 2.2 and 2.3. In [6], Theorem 2.2, A. Dow has proved that each weakly Lindelöf F'-space is an F-space.

Proposition 2.5. *Let X be an F'-space and $\{g_\mu : X \longrightarrow Y_\mu : \mu \in M\}$ be a family of open continuous mappings such that:*

- the space Y_μ is completely regular and submetrizable for each $\mu \in M$;

- for any infinite subset $A \subseteq X$ the set $g_\mu(A)$ is infinite for some $\mu \in M$.

Then any bounded subset of X is finite.

Corollary 2.1. *Let X be a subspace of the topological product $Y = \Pi\{Y_\mu : \mu \in M\}$ of completely regular spaces and for any non-empty countable subset $A \subseteq M$ the restriction of projection $p_A : X \longrightarrow Y_A \subseteq \Pi\{Y_\mu : \mu \in A\}$ of X onto $Y_A = p_A(X)$ is an open mapping. If Y is F'-space and any space Y_μ is submetrizable, then any bounded subset of X is finite.*

3 Mal'cev Spaces

Theorem 3.1. *If X is a Mal'cev F'-space, then any compact subset of X is finite.*

Proof. Assume that G is a Mal'cev F'-space. On G fix the Mal'cev operation $m : G^3 \longrightarrow G$. Fix a compact subset B of G. Assume that the set B is infinite.

Step 1. We put $G_0 = B$ and $G_{n+1} = m(G_n, G_n, G_n)$ for each $n \in \omega$. Let $G_\infty = \cup\{G_n : n \in \omega\}$. Then G_∞ is a Mal'cev space and the restriction $m|G_\infty$ is a Mal'cev operation on G_∞.

The set G_n is compact and $G_n \subseteq G_{n+1}$ for each $n \in \omega$. Thus the space G_∞ is σ-compact. By virtue of Proposition 2.4, G_∞ is an F-space.

Step 2. Since B is an infinite compact subset of G_∞, there exists a continuous function $f : G_\infty \longrightarrow [0,1]$ such that the set $f(B)$ is infinite. Since G_∞ is a σ-compact space, there exist a metrizable Mal'cev space M, a Mal'cev operation μ on M, a continuous mapping $g : G_\infty \longrightarrow M$ and a continuous mapping $h : M \longrightarrow [0,1]$ such that $f = h \circ g$ and $g(m(x,y,z)) = \mu(g(x), g(y), g(z))$ for all $x, y, z \in G_\infty$ (see [4]). By construction, the set $g(B)$ is infinite and compact.

Step 3. Let T_0 be the topology of the space M. Denote by T the quotient topology on M. Then (M,T) is a Mal'cev space, μ is a Mal'cev operation on (M,T) and the mapping g of G_∞ onto the space (M,T) is open (see [2]). Obviously, the restrictions of the topologies T and T_0 on $g(B)$ coincide. Hence for $Z = g^{-1}(g(B))$ the mapping $\psi : Z \longrightarrow g(B)$ is continuous and open. Since Z is a closed subspace of a normal σ-compact F-space G_∞, Z is an F-space too. By virtue of Proposition 2.1, $g(B)$ is an infinite compact metrizable F-space, a contradiction. The proof is complete. For the groups the next result was proved by A.V. Arhangel'skii [1].

Corollary 3.1. *If X is a Mal'cev extremally disconnected space, then any compact subset of X is finite.*

In the case of groups are true more general assertions.

Theorem 3.2. *Let H be a closed subgroup of a topological group G and*

$X = G/H$ be the quotient space. Assume that H is a uniform subgroup, or G is a group with quasi-invariant basis [3]. If a topological space X is an F'-space, then any bounded subset of X is finite.

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On hyperalgebras with conditions of homogenity

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Abstract

We study the topological hyperalgebras with conditions of homogeneity.

Keywords: topological universal algebras, totally bounded algebra, hyperalgebra.

1 Introduction

The theory of hyperalgebras arose at the dawn of modern algebra and has made a significant contribution in the solution of many important problems, to the development of combinatorics, to study color algebra and showed its utility in the study of groups, algebraic functions and rational fractions (see [6, 5, 4, 7].

Denote by $Com(X)$ the set of all non-empty compact subsets of a space X. A set-valued mapping $\theta : X \longrightarrow Y$ associate with each element x of a space X a non-empty subset $\theta(x)$ of a space Y. Let $\theta : X \longrightarrow Y$ be a set-valued mapping. The mapping θ is upper (lower) semicontinuous if the set $\theta^{-1}(H)$ is closed (open) in X for any closed (open) subset H of Y. The mapping θ is closed (open) if the set $\theta(W)$ is closed (open) in Y for any closed (open) subset W of X.

Let $\{E_n : n \in \mathbb{N} = \{0, 1, 2, 3, ...\}\}$ be a sequence of pairwise disjoint topological spaces. The discrete sum $E = \oplus\{E_n : n \in \mathbb{N}\}$ is the continuous signature of universal E-polyalgebras. A structure of an E-hyperalgebra on a non-empty space G is a family $\{e_{nG} : n \in \mathbb{N}\}$, where $e_{0G} : E_n \times G^n \to G$ is a single-valued mapping and $e_{nG} : E_n \times G^n \to G$ is a set-valued mapping for any $n \in \mathbb{N}$, $n \geq 1$. A topological universal hyperalgebra of the signature E or a topological E-hyperalgebra is a family $\{G, e_{nG} : n \in \mathbb{N}\}$, where G is a non-empty space, $e_{0G} : E_n \times G^n \to G$ is a single-valued continuous mapping and $e_{nG} : E_n \times G^n \to G$ is an uper semicontinuous compact-valued mapping for any $n \in \mathbb{N}$, $n \geq 1$. A mapping $\varphi : A \longrightarrow B$ of an E-hyperalgebra A into an E-hyperalgebra B is called: a weakly homomorphism if φ is a single-valued mapping and

$$\varphi(e_{nA}(u, x_1, x_2, ..., x_n) \subset e_{nB}(u, \varphi(x_1), \varphi(x_2), ..., \varphi(x_n))$$

for all $n \in \mathbb{N}$, $u \in E_n$ and $x_1, x_2, ..., x_n \in A$; a homomorphism if φ is a single-valued mapping and

$$\varphi(e_{nA}(u, x_1, x_2, ..., x_n) = e_{nB}(u, \varphi(x_1), \varphi(x_2), ..., \varphi(x_n))$$

for all $n \in \mathbb{N}$, $u \in E_n$ and $x_1, x_2, ..., x_n \in A$; an isomorphism if φ is a one-to-one homomorphism. Topological subhyperalgebras, weakly topological subhyperalgebras, Cartesian product of E-hyperalgebras are defined in the traditional way.

2 Polygroups and related algebras

Follows [1, 4, 7, 2, 3] we introduce the following notions.

A hypergroupoid is a hyperalgebra G with the unique binary operation $\{\cdot\}$. The element e is called: an identity of the hypergroupoid G if $e \cdot x = x \cdot e = x$ for any $x \in G$; a quasi-identity of the hypergroupoid G if $x \in e \cdot x \cap x \cdot e$ for any $x \in G$. Any identity is a quasi-identity. The quasi-identity is a result of a nulary operation.

A hipergroup is a hyperalgebra G with a unique binary operation $\{\cdot\}$, unary operation $\{^{-1}\}$ and a quasi-identity e for which: (HG1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in G$; (HG2) $x \in e \cdot x \cap x \cdot e$ for any $x \in G$; (HG3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $zy^{-1} \cdot x$ for all $x, y, z \in G$.

A hypergroup G is called a polygroup if: (PG1) e is an identity, i.e. $x \cdot e = e \cdot x = x$ for each $x \in G$; (PG2) $x \cdot x^{-1} = x^{-1} \cdot x = \{e\}$ for each $x \in G$.

If G is a topological space and a hypergroup and the operations $\{\cdot,^{-1}\}$ are compact-valued and upper semicontinuous, then G is a topological hypergroup. If G is a topological space and a polygroup and the operations $\{\cdot,^{-1}\}$ are compact-valued and upper semicontinuous, then G is a topological polygroup. Any topological group is a topological polygroup.

A topological homogeneous polyalgebra is a topological hyperalgebra G with two binary operation $p, q : G^2 \longrightarrow G$ for which: there exists a center $c \in G$ such that $c = p(x,x)$ for each $x \in G$; $\{x\} = p(x, q(x,y)) = q(x, p(x,y)$ for all $x, y \in G$.

An almost homeomorphism of a space X is a closed compact-valued uper-semicontinuous mapping $g : X \longrightarrow X$ for which there exists a closed compact-valued uper-semicontinuous mapping $f : X \longrightarrow X$ such that $g(f(x)) = f(g(x)) = x$ for each $x \in X$. In this case we put $f = g^{-1}$. Obviously that $g = f^{-1}$ and $(g^{-1})^{-1} = g$. A space X is called almost homogeneous if for any two points $a, b \in X$ there exists an almost homeomorphism $g : X \longrightarrow X$ such that $g(a) = b$. Any homogeneous space is almost homogeneous.

Proposition 3.3. *Let (G, p, q) be a topological homogeneous polyalgebra. For any $a \in G$ the set-valued mappings $P_a, Q_a : G \longrightarrow G$, where $P_a(x) = p(a,x)$ and $Q_a(x) = q(a,x)$ are compact-valued upper semicontinuous mappings with the following properties:*

(1) $P_a(a) = \{c\}$ and $Q_a(c) = \{a\}$.

(2) $P_a = Q_a^{-1}$, i.e. $P_a(Q(_a(x)) = Q_a(P(_a(x)) = \{x\}$ for each $x \in G$.

(c) P_a is an almost homeomorphism of the space G.

Corollary 3.4. *Any topological homogeneous polyalgebra is an almost homogeneous space.*

Proposition 3.5. *A topological polygroup $(G, e,^{-1}, \cdot)$ has a structure of a topological homogeneous polyalgebra.*

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On octahedral manifolds and their completions

Ion Gutsul

Abstract

We study octahedron manifolds and its completions in H^3.

Keywords: Hyperbolic space, manifolds, completions of manifolds.

1 Introduction

In the work [1] W.Thurston developed the theory of the completion of orientable hyperbolic 3-manifold. But he did not consider the completion of incomplete non-orientable hyperbolic manifolds. It is possible to develop the theory of the completion of such non-orientable manifolds, too, but the method of Thurston cannot be used in this case. For this case the hyperbolic space should be considered from the point of view of synthetic, i.e. Poincare models, or some other models of the hyperbolic space cannot be used.

Consider a complete non-compact hyperbolic 3-manifold M with finite volume. As it is shown in [2] it is rigid which means that two such manifolds with isomorphic fundamental groups are homeomorphic. If we begin to deform the manifold M it becomes incomplete but its completion is possible.

Our communication is devoted to both orientable and non-orientable manifolds obtained by the identification of faces of the hyperbolic octahedron with the vertices being infinitely removed.

In the hyperbolic space H^3 consider on octahedron with all the vertices on the absolute. The set of such octahedra forms a six-parameters family, i.e. depends on six continuons parameters. Because all the faces of these octahedra are congruent (as triangles with all the vertices on the absolute), there exist several schemes of the face identifications that yield manifolds, either orientable or non-orientable. These manifolds may differ one from another by the orientation (orientable or non-orientable) and by the number of cusps, i.e. subsets of the form $T^2 \times [0, \infty)$ (the produkt of a torus by a ray) which are orientable cusps and subset of the form $K^2 \times [0, \infty)$ (the product of a Kleins bottle by a ray) which are non-orientable cusps [3].

Label infinitely removed vertices of the octahedron with the numbers $1, 2, 3, 4, 5, 6$. For metric calculations we divide the octahedron into four simplexes:

$$T_1(1, 3, 2, 5); T_2(1, 3, 4, 5); T_3(1, 3, 4, 6); T_4(1, 3, 2, 6).$$

Let dihedral angles of these tetrahedra be:

$$T_1(\alpha_1, \beta_1, \gamma_1); T_2(\alpha_2, \beta_2, \gamma_2);$$

$$T_3(\alpha_3, \beta_3, \gamma_3); T_4(\alpha_4, \beta_4, \gamma_4).$$

Then the set of octahedra which are divided into these four simplexes forms a six-parametr family and satisfies the system of equations (1):

$$\alpha_1 + \beta_1 + \gamma_1 = \pi, \alpha_2 + \beta_2 + \gamma_2 = \pi;$$

$$\alpha_3 + \beta_3 + \gamma_3 = \pi; \alpha_4 + \beta_4 + \gamma_4 = \pi;$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi;$$

$$(sin\beta_1 sin\beta_2 sin\beta_3 sin\beta_4)/(sin\gamma_1 sin\gamma_2 sin\gamma_3 sin\gamma_4) = 1.$$

Identify faces of the octahedron by the following scheme:

$$(1, 2, 5)\varphi_1(3, 4, 5); (2, 3, 6)\varphi_2(4, 1, 6);$$

$$(2, 3, 5)\varphi_3(1, 6, 2); (1, 4, 5)\varphi_4(6, 3, 4)$$

Then the identifications of $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ yield an incomplete non-orientable manifold M_1 if dihedral angles of the simlexis satisfy the equations of system (1) and besides satisfy the following equations:

$$\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4;$$

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4;$$

$$(sin^2\gamma_1 sin\alpha_2 sin\beta_2 sin\beta_3 sin\alpha_4)/(sin\alpha_1 sin\beta_1 sin^2\gamma_2 sin\alpha_3 sin\beta_4) = 1$$

$$(sin^2\alpha_1 sin\beta_2 sin\gamma_2 sin\gamma_3 sin\beta_4)/(sin\beta_1 sin\gamma_1 sin^2\alpha_2 sin\beta_3 sin\gamma_4) = 1$$

$$(sin\beta_1 sin\alpha_2 sin\alpha_3 sin\beta_4)/(sin\alpha_1 sin\beta_2 sin\beta_3 sin\alpha_4) = 1$$

The manifold M_1 has one non-orientable cusp. We obtain the completion of the manifold M_1 if we odd to the equations one more equation

$$\alpha_4 - \beta_2 - \gamma_4 = \pi/k, k \geqslant 4.$$

As a result we obtain a countable series of non-orientable orbifolds. A fundamental polyhedron for these orbifolds is a truncated simplex, whose four faces are regular triangles with angles of $\alpha = \pi/(3k), k = 2, 3, ...$ and other four faces are hexagons with right angles.

Identify faces of the octahedron by following scheme:

$$(1, 2, 5)\varphi_1(3, 4, 5); (2, 3, 6)\varphi_2(4, 1, 6);$$

$$(2, 3, 5)\varphi_3(1, 2, 6); (1, 4, 5)\varphi_4(4, 3, 6),$$

we obtain an incomplete non-orientable manifolds M_2 if dihedral angles of the simplexes satisfy the equations of system (1) and the equations:

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4;$$

$$(sin\gamma_1 sin\beta_2 sin\beta_3 sin\gamma_4)/(sin\beta_1 sin\gamma_2 sin\gamma_3 sin\beta_4)\times$$

$$\times(sin^2\alpha_1 sin\beta_2 sin\gamma_2 sin\beta_3 sin\gamma_4)/(sin\beta_1 sin\gamma_1 sin^2\alpha_2 sin\gamma_3 sin\beta_4) = 1$$

$$(sin\beta_1 sin\gamma_2 sin^2\alpha_3 sin\beta_4 sin\gamma_4)/(sin\gamma_1 sin\beta_2 sin\beta_3 sin\gamma_3 sin^2\alpha_4) = 1$$

In this case 12 parameters of initial four simplex are related by ten equations, i.e. we have two free parameters. The manifold M_2 has two cusps and both are non-orientable. Therefore any completion of M_2 yields orbifolds. Moreover, we can obtain several countable series of orbifolds. If we demand that, besides the above ten equations, the following equations should be satisfied:

$$\beta_1 + \beta_2 = \beta_3 + \beta_4;$$

$$\beta_1 + \beta_3 - \beta_2 - \beta_4 = (2\pi)/k, k = 3, 4, ...$$

we obtain a countable series of orbifolds which are obtained from M_2 by completion on the cusp related to the vertices 5 and 6, whereas at vertices 1,2,3,4 there is a non-orientable cusp.

If we demand that, besides the above ten equations, the following equations should be satisfied:

$$\beta_1 + \beta_3 = \beta_2 + \beta_4;$$

$$\beta_3 + \beta_4 - \beta_1 - \beta_2 = (2\pi)/m, m = 3, 4, ...$$

then we obtain a countable series of orbifolds which are obtained from M_2 by completion on the cusp related to the vertices 1,2,3,4, where as at vertices 5,6 there is a non-orientable cusp.

Finally, if we complete the manifold M_2 on both cycles of vertices, then to the above ten equations the following equations should be added:

$$\beta_1 + \beta_3 - \beta_2 - \beta_4 = (2\pi)/k, k = 3, 4, ...;$$

$$\beta_3 + \beta_4 - \beta_1 - \beta_2 = (2\pi)/m, m = 3, 4, ...$$

Then we obtain a series of orbifolds which depends on two integer parameters. A fundamental polyhedron for these series of orbifolds is truncated tetrahedron. For faces of this polyhedron a triangles with one angle of $\pi/k, k = 3, 4, ...$, and two angles of $\pi/(2m), m = 3, 4, ...$ Other four faces of this polyhedron are hexagons with all right angles.

If we identify faces of the octahedron by the following scheme:

$$(1, 2, 5)\varphi_1(4, 3, 5); (2, 3, 5)\varphi_2(1, 4, 5);$$

$$(1, 2, 6)\varphi_3(4, 1, 6); (3, 4, 6)\varphi_4(2, 3, 6),$$

then we obtain an incomplete non-orientable 3-manifold with three cusps. One of this cusps is orientable, the two other cusps are non-orientable. To the equations of the octahedron the following equations should be added:

$$(sin\beta_1 sin\beta_2)/(sin\gamma_1 sin\gamma_2) = 1$$

$$sin\beta_1 sin\beta_2 sin\alpha_3 sin\alpha_4)/(sin\alpha_1 sin\alpha_2 sin\gamma_3 sin\gamma_4) = 1$$

If to the obtained equations we add the equations:

$$\alpha_3 = \alpha_4, \beta_3 = \beta_4,$$

then both non-orientable cusps will be complete.

For the completion of orientable cusp at vertex 5 we consider a horosphere S centred at 5. On this horosphere we obtain a similarity symmetry group that is induced by the motions φ_1 and φ_2. In order this group be discrete it should be generated by two spiral rotations f_1 and f_2 which are related by the equations:

$$m \times \psi_1 + n \times \psi_2 = 2\pi; k_1^m = k_2^n,$$

where m and n are natural coprime numbers and $m + n \geqslant 5$. In this equations ψ_1 and ψ_2 are rotation angles, k_1 and k_2 are similarity coefficients of the spiral rotations f_1 and f_2. Then for the completion of the cusp to the above system the following equations should be added:

$$m(\beta_1 - \beta_2) + n(\gamma_2 - \gamma_1) = 2\pi;$$

$$((sin\gamma_1 sin\alpha_2)/(sin\alpha_1 sin\gamma_2))^m = ((sin\alpha_1 sin\beta_2)/(sin\beta_1 sin\alpha_2))^n,$$

where m and ;n are natural co-prime numbers and $m+n \geqslant 5$. Varying the numbers m and n we obtain a countable series of non-orientable noncompact hyperbolic 3-manifolds $M_m n$, and the volumes of these manifolds are bounded by the volume of a regular hyperbolic octahedron with all the vertices being on the absolute.

If we identify faces of the octahedron by the following scheme:

$$(1,2,5)\varphi_1(2,3,5);(1,2,6)\varphi_2(4,3,5);$$

$$(1,4,5)\varphi_3(2,3,6);(1,4,6)\varphi_4(4,3,6),$$

we obtain an incomplete non-orientable 3-manifolds with two cusps. One of these cusps is orientable, the other cusp is non-orientable. In this case to the equations of octahedron the following three equations should be added:

$$\alpha_2 = \alpha_4;$$
$$(sin^2\gamma_1 sin\gamma_2 sin\gamma_4)/(sin^2\beta_1 sin\beta_2 sin\beta_4) = 1;$$
$$(sin\alpha_1 sin\beta_2 sin\alpha_3 sin\beta_4)/(sin\gamma_1 sin\alpha_2 sin\gamma_3 sin\alpha_4) = 1.$$

Then reasoning as in the previous case we obtain a countable series of non-orientable noncompact manifolds $P_m n$ with one non-orientable cusp whose volumes are bounded by the volume of a regular hyperbolic octahedron with all the vertices on the absolute.

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On polyhedra in H^3 with right dihedral angles

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Abstract

We study unbounded polyhedra with finite volume in H^3 whose all dihedral angles are equal to $\pi/2$.

Keywords: Hyperbolic space, polyhedra.

1 Introduction

The problem of bounded polyhedra in the space H^3 with all right dihedral angles has been solved in the work [1]. The present communication is aimed at the question of unbounded polyhedra with finite volume in H^3 whose all dihedral angles are equal to $\pi/2$. It is clear that such polyhedra tile H^3 face-to-face and tile-transitive.

We do reasoning from the point of view of synthetic geometry. It is clear that to each proper vertex of a polyhedron with all right angles exactly three edges are adjacent while to each infinitely remove vertex of such a polyhedron exactly four edges are adjacent. Such a polyhedron can have k-gons as two-dimensional faces with proper vertices for $k \geq 5$ and l-gons as two-dimensional faces with all infinitely remote vertices for $l \geqslant 3$.

One of the simplest unbounded polyhedra with all right angles can be obtained by "glueing" together of two halves of a simplex with all infinitely remote vertices and dihedral angles $(\pi/2, \pi/4, \pi/4$. As a result we obtain a polyhedron with two proper vertices and three infinitely remote vertices. All its faces are triangles with one right angle and two zero angles. This polyhedron seems to have the minimum volume among polyhedra of such kind.

2 Some results

Now introduce one concept we need for further reasoning.

A spherical sack is a closed surface diffeomorphic to the two-dimensional sphere. Any chord situated inside the surface and joining any two points of the surface will be called a diameter of the spherical sack. It is clear that diameters will have different lengths. We say that we blow the spherical sack meaning some diameters increase in length but not necessarily all of them, some diameters may decrease in length.But it is obligatory that the volume of the space inside the spherical sack should increase.

Now we explain our method of contraction, of a bounded edge of a polyhedron into a point of the absolute (i.e. into an infinitely remote vertex). The method includes three stages. At the first stage we obtain a polyhedron K with all the vertices being on the absolute (i.e. each proper vertex becomes infinitely remote). At the second stage we remove the vertices beyond the absolute. So in the obtained polyhedron edges adjacent to a vertex form a hyperbolic bundle of straight lines, and we cut the bundle by a plane orthogonal to the straight lines of the bundle. As a result we obtain a new polyhedron P. In the polyhedron P the number of faces increases, because new faces appear corresponding to vertices of the polyhedron K and orthogonal to faces of the polyhedron P inherited from the polyhedron K. At the third stage we obtain a polyhedron M when all the edges of the polyhedron P inherited from the polyhedron K are contracted into points of the absolute, i.e. they become infinitely remote vertices of the polyhedron M and the polyhedron M has all dihedral angles equal to $\pi/2$.

Theorem 1. *Let K be a bounded polyhedron in the hyperbolic space H^3. Then in H^3 there exists a corresponding to K unbounded polyhedron P of finite volume with all vertices being infinitely remote and with all right dihedral angles which has the following structure:* 1) *the number of infinitely remote vertices of the polyhedron P is equal to the number of edges of the polyhedron K;* 2) *faces of the polyhedron P belongs to two types: polygons corresponding to faces of the polyhedron K and polygons corresponding to polyhedral vertices of the polyhedron K.*

Consider the simplest bounded polyhedron in H^3, a tetrahedron. Inscribe in the tetrahedron a spherical sack and begin to blow it. The tetrahedron will decrease in its volume and its dihedral angles will decrease. Continue to blow spherical sack and at some moment all the vertices of the tetrahedron become infinitely remote, then they go beyond the absolute. Then the edges adjacent to a vertex form a hyperbolic bundle. For each vertex cut the bundle by a plane orthogonal to the edges. As a result we obtain a truncated tetrahedron whose four faces are triangles and four faces are hexagons with all inner angles equal to $\Pi/2$. If we continue to blow the spherical sack, dihedral angles at "old" edges will decrease as well as "old" edges will decrease,where as dihedral angles at "new" edges will become right and "new" edges will increase in length. If go to limit, "old" edges will contract into points of the absolute, but "new" edges will become straight lines. The truncated tetrahedron will become a regular octahedron with all vertices being infinitely remote and with right dihedral angles.

The prof of Theorem 1 reduces to the three above-mentioned stages. At the first stage we inscribe in to the polyhedron K a spherical sack and begin to blow it. The initial polyhedron will increase in volume, its dihedral angles will decrease, edges will increase. We continue the process until all the vertices of the polyhedron K become infinitely remote and edge segments become straight lines. Blowing further the spherical sack we achieve the case when the edges of the polyhedron K adjacent to each its vertex will form a hyperbolic bundle of straight lines. Then for each such hyperbolic bundle there exists a plane orthogonal to all the straight lines of the bundle. Cut by such planes the parts that go beyond the absolute and obtain a new polyhedron S. Faces of the polyhedron S correspond to polyhedral vertices of the polyhedron K as well as to faces of K truncated at vertices. So the number of faces of the polyhedron S is equal to the number of faces of the polyhedron K plus the number of vertices of K. Dihedral angles at "new" faces of the polyhedron S is equal to the number of faces of the polyhedron K plus the number of vertices of K. Dihedral angles at "new" faces of the polyhedron S will be right, the polyhedron S will be bounded. Continuing to blow the spherical sack we will obtain polyhedra analogous to S, but the vertices of "new" and "old" faces will tend to the absolute, the lengths of edges of "new" faces will increase, and dihedral angles at these faces will remain right. As to the "old" edges of the polyhedron S, their "lengths" and dihedral angles at them will decrease. Finally, continuing to blow the spherical sack we will achieve the case when all the "old" edges of the polyhedron S will contract into points of the absolute, and we will obtain a polyhedron P with all right angles.

Theorem 2. *Let K be a polyhedron of finite volume with proper and infinitely remote vertices in the hyperbolic space H^3. Then in H^3 there exists a corresponding to K polyhedron P of finite volume with all the vertices being on the absolute and with all right dihedral angles which has the following structure:* 1) *the number of infinitely remote vertices of the polyhedron P is equal to the number of edges of the polyhedron K;* 2) *faces of the polyhedron P belong to two types: polygons corresponding to faces of the polyhedron K and polygons corresponding to polyhedral vertices of the polyhedron K (both proper and infinitely remote).*

Theorem 3. *Let K be a polyhedron of finite volume with all the vertices being infinitely remote in the hyperbolic space H^3. Then in H^3 there exists a corresponding to K polyhedron P of finite volume with all the vertices being on the absolute and with all right dihedral angles which has the following structure:* 1) *the number of infinitely remote vertices of the polyhedron P is equal to the number of edges of the polyhedron K;* 2) *faces of the polyhedron P belong to two types: polygons corresponding to faces of the polyhedron K and polygons corresponding to polyhedral vertices of the polyhedron K.*

The prof of Theorems 2 and 3 can be done analogously to the prof of Theorem 1.

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The First Fundamental Theorem for Similarity Groups in D^3 and Application to Integral B-Splines

Muhsin Incesu, Osman Gürsoy

Abstract

In this paper we stated the First Fundamental Theorem for Similarity Group in three dimensional space of dual vectors $S(3, D)$ and its main subgroup $LS(3, D)$. Then applied the results of this theorem to Spatial NURBS curves and surfaces and studies results by real spaces.

Keywords: FFT, B-Splines, space of dual vectors, similarity groups

1 Introduction

A **dual number** A is defined as $A = a + \epsilon a^*$, where a and $a^* \in R$ and $\epsilon^2 = 0$ ($\epsilon \neq 0$). The set of all dual numbers is denoted by D. Similarly a dual vector X is defined as $X = x + \epsilon x^*$, where x and $x^* \in R^3$ and $\epsilon^2 = 0$. The set of all dual vectors is denoted by D^3.

Let D^3 be a three dimensional dual Euclidean space then the transformation $F : D^3 \mapsto D^3$ such that $\|F(x) - F(y)\| = \lambda \|x - y\|$ is called a **similarity transformation** if there exist a Λ dual number such that $\Lambda = \lambda + \epsilon\lambda^*$ and $\lambda > 0$ for every $x, y \in D^3$.

The group of all the orthogonal transformations defined in D^3 is denoted by $O(3, D)$. The group of all the linear similarity transformations defined in D^3 is denoted by $LS(3, D)$. The group of all the similarity transformations (including translations) defined in D^3 is denoted by $S(3, D)$.

2 First Fundamental Theorem for Similarity Groups

Let $D\left[x^{(1)}, x^{(2)}, ..., x^{(m)}\right]$ be a ring of polynomials for m vector variables $x^{(1)}, x^{(2)}, ..., x^{(m)}$ in D^3 over the field D and a transformation group G be given. Then the algebra of $G-$invariant polynomials for m vector variables $x^{(1)}, x^{(2)}, ..., x^{(m)}$ in D^3 over the field D is denoted by $D\left[x^{(1)}, x^{(2)}, ..., x^{(m)}\right]^G$ and the field of $G-$invariant rational functions for m vector variables $x^{(1)}, x^{(2)}, ..., x^{(m)}$ in D^3 over the field D is denoted by $D\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right)^G$.
Theorem (FFT for $LS(3,D)$): Let $x^{(1)}, x^{(2)}, ..., x^{(m)}$ be m vector variables different from zero in D^3. For $i, j = 1, 2, ..., m$ the generator system of the field $D\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right)^{LS(3,D)}$ is as follows

$$\frac{\left\langle x^{(i)}, x^{(j)}\right\rangle}{\left\langle x^{(1)}, x^{(1)}\right\rangle}, i \leq j, \tag{1}$$

Theorem : If any rational function $f\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right)$ with m vector variables is $S(3,D)-$ invariant then the rational function g defined by

$$g\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right) = f\ \left(0, x^{(2)} - x^{(1)}, x^{(3)} - x^{(1)}, ..., x^{(m)} - x^{(1)}\right) \tag{2}$$

is $LS(3,D)-$ invariant. On the contrary Let the function $f\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right)$ be stated as

$$f\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right) = h\left(x^{(2)} - x^{(1)}, x^{(3)} - x^{(1)}, ..., x^{(m)} - x^{(1)}\right) \tag{3}$$

then if the function h is $LS(3,D)-$ invariant then the function f is $S(3,D)-$ invariant.
Theorem :(FFT for $S(3,D)$) Let $x^{(1)}, x^{(2)}, ..., x^{(m)}$ be m vector variables different from each other in D^3. For $i, j = 1, 2, ..., m$ the generator system of the field $D\left(x^{(1)}, x^{(2)}, ..., x^{(m)}\right)^{S(3,D)}$ is as follows

$$\frac{\left\langle x^{(i)} - x^{(1)}, x^{(j)} - x^{(1)} \right\rangle}{\left\langle x^{(2)} - x^{(1)}, x^{(2)} - x^{(1)} \right\rangle}, i \leq j, \tag{4}$$

3 Non Uniform B-Spines

Definition: The B- Spline basis functions of degree k, denoted $N_{i,k}(t)$, defined by the knot vector $t_0, t_1, ..., t_m$ are defined recursively as follows

$$N_{i,0}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t) \tag{6}$$

for $i = 1, 2, ..., n$.

Definition: The B- Spline curve of degree k on the interval $[a, b]$ with control points $b_0, b_1, ..., b_n$ and knots $t_0, t_1, ..., t_m$ such that $t_i \leq t_{i+1}$ (for $i = 0, 1, ...m - 1$) and $[a, b] = [t_k, t_{m-k}]$ $(0 \leq k \leq m)$ is defined by

$$B(t) = \sum_{i=0}^{n} b_i N_{i,k}(t)$$

where $N_{i,k}(t)$ are the B-Spline basis functions of degree k. [3]

Theorem: Let $B(t)$ be the B-Spline curve of degree k on the interval $[a, b]$ with control points $b_0, b_1, ..., b_n$ and knots $t_0, t_1, ..., t_m$ properties and T be an Affine transformation Then
$T(B(t)) = \sum_{i=0}^{n} T(b_i) N_{i,k}(t)$ for $t \in [t_r, t_{r+1})$
is satisfied.[3]

Theorem: Let $B(t)$ be the B-Spline curve of degree k on the interval $[a, b]$ with control points $b_0, b_1, ..., b_n$ and knots $t_0, t_1, ..., t_m$ Then any $LS(3, D)$-invariant property of $B(t)$ can be stated as the function of the elements of these system

$$\frac{\langle b_i, b_j \rangle}{\langle b_1, b_1 \rangle}, i \leq j, \tag{7}$$

Theorem: Let $B(t)$ be the B-Spline curve of degree k on the interval $[a, b]$ with control points $b_0, b_1, ..., b_n$ and knots $t_0, t_1, ..., t_m$ Then any $S(3, D)$-invariant property of $B(t)$ can be stated as the function of the elements of these system

$$\frac{\langle b_i - b_1, b_j - b_1 \rangle}{\langle b_2 - b_1, b_2 - b_1 \rangle}, i \leq j, \tag{8}$$

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The generalized symmetry of the geometrical figures regularly weighted by scalar tasks

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Abstract

In this paper are studied the mixed transformations of the geometrical figures regularly weighted by "physical" scalar tasks. Are determined the conditions in which one mixed transformation is or exactly transformation of P-symmetry, or exactly transformation of W_p-symmetry. Some crystallographic groups of W_p-symmetry are analysed.

Keywords: groups, symmetry, generalized symmetries.

1 Introduction

The theory of symmetry of the real crystals gives rise to new generalizations of classical symmetry: the Shubnikov's antisymmetry, the multiple antisymmetry, the Belov's colour symmetry, the Zamorzaev's P-symetry, the cryptosymmetry, e.t.c. We shall discuss briefly the essence of the generalized symmetry of the geometrical figures regularly weighted by scalar tasks.

2 Mixed transformations of figures regularly weighted by scalar tasks

Let us have discrete group of symmetry G of geometrical figure F from geometric space S of constant curvature and finite set $N = \{1, 2, ..., m\}$ of "indexes", which mean a non-geometrical feature. Let each "index"

r from the set N have a scalar nature (temperature, density, color). On fix a certain transitive group P of permutations over N. The decomposition $\{S_i\}$ of space S within respect to the group G always is regular. We will note with the symbol F_i the intersection of geometric figure F with the fundamental domain S_i of the decomposition $\{S_i\}$ of the space S. Ascribe to each interior point M of F_i the same "indexes" $r_1, r_2, , ..., r_k$, from the set N, where k is a divisor of $m = |N|$. We obtain one regularly weighted figure $F^{(N)}$ with summary scalar load N.

The mixed transformation $\tilde{g}$ of the "indexed" figure $F^{(N)}$ is composed of two independent components: $\tilde{g} = gw$, where g is pure geometrical isometric transformation and w is certain complex rule which describes the transformation of the "indexes" $r \in N$, ascribed to the interior points M of certain domain F_i.

If the rule w is the same for every "indexed" point of space, then the mixed transformation $\tilde{g}$ is exactly a transformation of P-symmetry [1]. The set of transformations of P-symmetry of any geometric figure F well-balanced by scalar "physical" load N forms a group, where is subgroup of the direct product of the group P with generating group G.

The theory of P-symmetry groups, inclusive the methods of deriving the groups of different tips from given groups P and G was elaborated and developed by Zamorzaev's geometrical school from Chisinau [1-3].

The "indexes" r_i and r_j, ascribed to the points which belong to distinct domains F_i and F_j, are transformed, in general, by different permutations p_i and p_j. The permutations p_i and p_j are from given transitive group of permutations P over N. In this case the rule w is composed exactly from $|G|$ components-permutations $p \in P$. In conditions of this case the transformation $\tilde{g} = gw$ is exactly a transformation of W_p-symmetry [3,4]. The set of transformations of W_p-symmetry of the given "indexed" figure $F^{(N)}$ forms a group, which fulfil certain conditions of classification, where is subgroup of the left standard Cartaisian wreath product of initial group P with group G.

3 Some properties of W_p-symmetry groups

Let $G^{(W_p)}$ be a group of W_p-symmetry with initial group P, generating group G and subset $W' = \{w|g^{(w)} \in G^{(W_p)}\} \subseteq W$, with the symmetry subgroup $H = G^{(W_p)} \cap G$ and the subgroup $V = G^{(W_p)} \cap W = G^{(W_p)} \cap W'$ of W-identical transformations. The group $G^{(W_p)}$ is called major if $w_0 < V = W' = W$. If W' is a non-trivial subgroup of W, then the group $G^{(W_p)}$ is called W'-semi-major or W'-semi-minor according to the cases when $w_0 < V = W'$ or $w_0 = V < W'$. If $W' \subset W$, but W' is not a group, $G^{(W_p)}$ is called W'-pseudo-minor when $w_0 = V \subset W'$.

Any major group of W_p-symmetry with the finite groups G and P it is construct in shape of the left standard direct wreath product of group G with initial group P, accompanied with a fixed isomorphism $\varphi : G \to AutW$ by the rule $\varphi(g) = \overleftharpoon{g}$, where $\overleftharpoon{g}: w \mapsto w^g$.

Any W'-semi-major finite group of W_p-symmetry with initial group P and generating group G can be derived from G and P by the following steps: 1) we construct the direct product W of isomorphic copies of the group P which are indexed by elements of G; 2) we find in W so non trivial subgroups W' wich verify the conditions $\overleftharpoon{g}\ (W')W' = W'$, for each g from group G; 3) we combine pairwise each g of group G with each w of subgroup W'; 4) we introduce into the set of all these pairs the operation $g_iw_i \circ g_jw_j = g_kw_k$, where $g_k = g_ig_j$, $w_k = w_i^{g_j}w_j$ and $w_i^{g_j}(g_s) = w_i(g_jg_s)$.

Any W'-semi-minor (respectively, pseudo-minor) finite group of W_p-symmetry with initial group P and generating group G can be derived from G and P by the following steps: 1) we construct the direct product W of isomorphic copies of the group P which are indexed by elements of G; 2) we find in W so non trivial subgroups W' (respectively, the subset W' with unit, which is not a subgroup) wich verify the condition $\overleftharpoon{g}\ (W')W' = W'$, for each g from group G; 3) we construct an exact natural left quasi-homomorphism μ with the kernel H of the group G onto the subgroup W' (respectively, onto the subset W' with unit, which is not a subgroup) by the rule $\mu(Hg) = w$; 4) we combine pairwise each g of class Hg with $w = \mu(Hg)$; 5) we introduce into the set of all these pairs the operation $g_i w_i \circ g_j w_j = g_k w_k$, where $g_k = g_i g_j$, $w_k = w_i^{g_j} w_j$ and $w_i^{g_j}(g_s) = w_i(g_j g_s)$.

4 Concrete results

From the crystallographic point generating groups G and group P of permutations ($P \cong C_2$), we obtained 32 major groups of W_p-symmetry. From the netrivial cyclical crystallographic punctual groups G and group P ($P \cong C_2$), we obtained: 9 major groups, 20 W'-semi-major groups, 10 semi-minor groups and 20 pseudo-minor groups of W_p-symmetry.

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On Set-Valued Periodic Functions on Topological Spaces

Dorin I. Pavel

Abstract

In this paper we study general properties of almost periodic set-valued functions on a given topological space.

Keywords: Pompeiu-Hausdorff distance, almost periodicity, set-valued function.

1 Introduction

Let A and B be two non-empty subsets of a metric space (G, d). We define their Pompeiu-Hausdorff distance $d_P(A, B)$ by $d_P(A, B) = max\{sup\{inf\{d(x, y) : y \in B)\} : x \in A\}, sup\{inf\{d(x, y) : x \in A)\} : y \in B\}\}$.

Denote by d the Euclidean distance on the space $\mathbb{R}$ of reals and $B(\mathbb{R})$ the space of all non-empty bounded subsets of $\mathbb{R}$ with the Pompeiu-Hausdorff distance $d_P(A, B)$.

The space $(B(\mathbb{R}), d_P)$ is a complete pseudometric space with the properties: $d_P(A, B) = 0$ if and only if $clA = clB$; $d_P(A, B) = d_P(B, A)$; $d_P(A, C) \leq d_P(A, B) + d_P(B, C)$.

Fix a topological space G. By $T(G)$ denote the family of all single-valued continuous mappings of G into G. Relatively to the operation of composition, the set $T(G)$ is a monoid (a semigroup with unity).

A single-valued $\varphi : G \longrightarrow B(\mathbb{R})$ is called a set-valued function on G. For any two set-valued functions $\varphi, \psi : G \longrightarrow B(\mathbb{R})$ and $t \in \mathbb{R}$ are determined the distance $\rho(\varphi, \psi) = sup\{d_P(\varphi(x), \psi(x)) : x \in G\}$ and the set-valued functions $\varphi + \psi$, $\varphi \cdot \psi$, $-\varphi$, $\varphi \cup \psi$, where $(\varphi \cup \psi)(x) =$

$(\varphi(x) \cup \psi(x)$, and $t\varphi$. We put $(\varphi \circ f)(x) = \varphi(f(x))$ for all $f \in T(G)$, $\varphi \in SF(G)$ and $x \in G$.

If $0 \notin \varphi(x)$ for each $x \in G$, then is determined the set-valued functions φ^{-1}.

For any non-empty compact subset F of $\mathbb{R}$ is determined the constant set-valued function k_F, where $k_F(x) = F$ for each $x \in G$. If $t \in \mathbb{R}$, we put $k_t = k_{\{t\}}$. The constant function k_F is continuous and closed.

Let $SF(G)$ be the family of all set-valued functions on G. The space $(SF(G), \rho)$ is a complete pseudometric space with the properties: (1) $\rho(\varphi, \psi) = 0$ if and only if $cl\varphi(x) = cl\psi(x)$ for each $x \in G$; (2)$\rho(\varphi, \psi) = \rho(\psi, \varphi)$; (3) $\rho(\varphi, \theta) \leq \rho(\varphi, \psi) + \rho(\psi, \theta)$. Denote by $SF^*(G)$ the family of all bounded set-valued functions on G, by $SF_c(G)$ the family of all compact-valued functions on G and $SF^*_c(G) = SF_c(G) \cap SF^*(G)$.

The space $(SF_c(G), \rho)$ is a complete metric space and the sets $SF^*(G)$ and $SF_c(G)$ are closed in $SF(G)$.

2 Algebras of set-valued functions

Fix a space G. A family $\mathcal{L} \subset SF(G)$ is called an m-group of set-valued functions if: $k_0 \in \mathcal{L}$; $-\varphi, \varphi + \psi \in \mathcal{L}$ for any $\varphi, \psi \in \mathcal{L}$. A family $\mathcal{L} \subset SF(G)$ is called an m-ring of set-valued functions if: $k_1 \in \mathcal{L}$ and $\mathcal{L} \subset SF(G)$ is an m-group; $t\varphi, \varphi \cdot \psi \in \mathcal{L}$ for any $\varphi, \psi \in \mathcal{L}$. A family $\mathcal{L} \subset SF(G)$ is called an m-algebra of set-valued functions if: $\mathcal{L} \subset SF(G)$ is an m-ring; $\varphi^{-1} \in \mathcal{L}$ provided $\varphi \in \mathcal{L}$ and $0 \notin cl_{\mathbb{R}}\varphi(x)$ for each $x \in G$.

Obviously, $SF(G)$, $SF^*(G)$ and $SF_c(G)$ are m-algebras of set-valued functions. Moreover, the family $C(G)$ of all continuous single-valued functions $f : G \longrightarrow \mathbb{R}$ is an m-algebra of set-valued functions. The intersection of any family of m-algebras (m-rings, m-groups) is an m-algebra (m-ring, m-group).

Theorem 2.1. *The family $LSC(G)$ of all lower semicontinuous functions is an m-algebra of set-valued functions. Moreover, we have $\varphi \circ f, \varphi \cup \psi \in LSC(G)$ for all $\varphi, \psi \in LSC(G)$ and $f \in T(G)$.*

Corollary 2.2. *The family $LSC^*(G)$ of all bounded lower semicontinuous functions and the family $LSC_c(G)$ of all compact-valued lower semicontinuous functions are m-algebras of set-valued functions. Moreover, we have $\varphi \circ f, \varphi \cup \psi \in LSC^*(G)$ for all $\varphi, \psi \in LSC^*(G)$ and $f \in T(G)$. Similarly, we have $\varphi \circ f, \varphi \cup \psi \in LSC_c(G)$ for all $\varphi, \psi \in LSC_c(G)$ and $f \in T(G)$.*

Theorem 2.3. *The family $USC(G)$ of all upper semicontinuous functions is an m-algebra of set-valued functions. Moreover, we have $\varphi \circ f, \varphi \cup \psi \in USC(G)$ for all $\varphi, \psi \in USC(G)$ and $f \in T(G)$.*

Corollary 2.4. *The family $USC^*(G)$ of all bounded upper semicontinuous functions and the family $USC_c(G)$ of all compact-valued upper semicontinuous functions are m-algebras of set-valued functions. Moreover, we have $\varphi \circ f, \varphi \cup \psi \in USC^*(G)$ for all $\varphi, \psi \in USC^*(G)$ and $f \in T(G)$. Similarly, we have $\varphi \circ f, \varphi \cup \psi \in USC_c(G)$ for all $\varphi, \psi \in USC_c(G)$ and $f \in T(G)$.*

3 Almost periodic set-valued functions

Fix a topological space G. On $SF(G)$ we consider the topology generated by the distance ρ. If $\varphi \in SF(G)$ and $f \in T(G)$, then $\varphi_f = \varphi \circ f$ and $\varphi_f(x) = \varphi(f(x))$ for any $x \in G$). Evidently, $\varphi_f \in SF(G)$.

Fix a submonoid P of the monoid $T(G)$. We say that P is a monoid of continuous translations of G. The set P is called a transitive set of translations of G if for any two points $x, y \in G$ there exists $f \in P$ such that $f(x) = y$.

For any function $\varphi \in C(G)$ we put $P(\varphi) = \{\varphi_f : f \in P\}$.

Remark 3.1. *The following assertions are true: (1) If $\varphi \in SF^*(G)$, then $P(\varphi) \subseteq SF^*(G)$. (2) If $\varphi \in SF_c(G)$, then $P(\varphi) \subseteq SF_c(G)$. (3) If $\varphi \in LSC(G)$, then $P(\varphi) \subseteq LSC(G)$. (4) If $\varphi \in USC(G)$, then $P(\varphi) \subseteq USC(G)$.*

Definition 3.2. *A function $\varphi \in SF(G)$ is called a P-periodic function on a space G if $\varphi \in SF^*(G)$ and the closure $\bar{P}(\varphi)$ of the set $P(\varphi)$ in the space $SF(G)$ is a compact set.*

Denote by P-$ap_s(G)$ the subspace of all P-periodic set-valued functions on the space G.

If the monoid P is finite, then P-$ap_s(G) = SF^*(G)$.

Theorem 3.3. *Let P be a monoid of continuous translations of G. Then P-$ap_s(G)$ has the following properties:*

1. P-$ap_s(G)$ is an m-algebra of set-valued functions on the space G.

2. If $\varphi \in SF(G)$ and $\bar{\varphi}(x) = cl_{\mathbb{R}}\varphi(x)$ for any $x \in G$, then $\varphi \in P$-$ap_s(G)$ if and only if $\bar{\varphi} \in P$-$ap_s(G)$.

3. P-$ap_s(G)$ is a closed subspace of the complete pseudometric space $SF(G)$.

4. If $\varphi \in SF(G)$ is a constant set-valued function, then $\varphi \in P$-$ap_s(G)$.

Proof. Assertion 1 follows from the following facts:

- if A is a compact subset of $\mathbb{R}$ and $0 \notin A$, then the set A^{-1} is compact;

- if A and B are compact subsets of $\mathbb{R}$, then the sets $A \cup B$, $A + B$ and $A \cdot B$ are compact.

Assertion 2 follows from the equality $P(\bar{\varphi}) = P(\bar{\bar{\varphi}})$ for any $\varphi \in SF(G)$. Assertion 3 follows from the completeness of the Pompeiu-Hausdorff distance $d_P(A, B)$. Assertion 4 is obvious.

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String Theory Phenomenology Without Compactification or Localisation

Alexandru Popa

Abstract

In the paper the model of space is proposed that is suitable for the string theory. The model is globally isotropic and homogeneous. The advantage of described space is acceptable phenomenology without compactification or localisation.

Keywords: homogeneous space, string theory, M-theory, compactification, localisation.

1 Introduction

As known [1], the string theory, or its further development, superstring theory and M-theory is at present the only serious candidate to "Theory of Everything" in physics. The main its problem however is the phenomenology problem, that is, all variants require the space with many dimensions: 26, 10 or 11, while the observable space–time is 4 dimensional. Two main ideas that were proposed to solve the phenomenology problem of string theory are *compactification* or *localisation*.

Both ways look like artificially introduced tricks to match to the appearance rather than the physical reality, as none of these ways constructs isotropic and homogeneous space with observable properties, as it is to expect from isotropically and homogeneously acting physical principles and laws. This paper proposes another geometric approach, that constructs a homogeneous space with necessary dimension and with observable 4-dimensional space–time.

2 Preliminaries

The model of homogeneous spaces was introduced in [2]. The natural space metric was introduced with aim of generalized trigonometric functions ($k \in \{-1, 0, 1\}$):

$$C(x) = C(x,k) = \sum_{n=0}^{\infty} (-k)^n \frac{x^{2n}}{(2n)!} = \begin{cases} \cos x, & k = 1, \\ 1, & k = 0, \\ \cosh x, & k = -1; \end{cases}$$

$$S(x) = S(x,k) = \sum_{n=0}^{\infty} (-k)^n \frac{x^{2n+1}}{(2n+1)!} = \begin{cases} \sin x, & k = 1, \\ x, & k = 0, \\ \sinh x, & k = -1. \end{cases}$$

It was also mentioned [3] that not natural metric parameters can be embedded in definition of generalized trigonometric functions.

3 Main result

Construct the functions $Cr(x), Sr(x)$, for some $r \in \mathbb{R}$. They are related to generalized trigonometric functions ($r = kp^2$, $k = \operatorname{sign} r, p = \sqrt{|r|}$):

$$Cr(x) = \sum_{n=0}^{\infty} (-r)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-k)^n \frac{(px)^{2n}}{(2n)!} = C(px), \tag{1}$$

$$Sr(x) = \sum_{n=0}^{\infty} (-r)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{p} \sum_{n=0}^{\infty} (-k)^n \frac{(px)^{2n+1}}{(2n+1)!} = \frac{1}{p} S(px). \tag{2}$$

So the equation $C^2(px) + kS^2(px) = 1$ becomes:

$$Cr^2(x) + kp^2 Sr^2(x) = Cr^2(x) + r\, Sr^2(x) = 1, \text{ for some } r \in \mathbb{R}. \tag{3}$$

Because the functions $Cr(x), Sr(x)$ are expressible via $C(x), S(x)$, all the theory can be constructed on them, rather than on generalized trigonometric functions. Namely, in virtue of (3), the theorem about characteristic multiplication, $K_{i,j} = \prod_{l=i}^{j-1} k_l$ becomes $R_{i,j} = \prod_{l=i}^{j-1} r_l$.

4 Proper phenomenology

Let $\mathbb{B}^n = \{k_1, ..., k_n\}$ and n depends on concrete string or M-theory. We can assume that:

- $r_1 \approx \frac{1}{\hbar}$ is a large positive number depending on Planck's constant. It ensures microcosmic closed lengths across the first dimension.

- $r_2, ..., r_{n-4} \approx 1$ are positive numbers of order of magnitude of 1. It ensures microcosmic closed lengths also in $\mathbb{B}^{n-4}$ subspace.

- $r_{n-3} \approx -\Lambda$ depends on cosmological constant. It is small number that ensures opening of the further dimensions to have macroscopic lengths. The following cases are possible:

  $r_{n-3} < 0$ models de Sitter space–time,

  $r_{n-3} = 0$ models Minkowski space–time,

  $r_{n-3} > 0$ models anti de Sitter space–time.

- $r_{n-2} = -\frac{1}{c}$, where c is the speed of light. It ensures the connection between space and time in Einstein's relativity. Optionally:

  $r_{n-2} = 0$ models the Galilean space and time connection.

- $r_{n-1} = r_n = 1$ ensures 3-dimensional locally Euclidean space.

This model is similar to $\mathbb{S}^{n-4} \times \mathbb{M}^4$ space used in some string theories, but it is globally homogeneous space. It has microscopic $n - 4$ dimensions, phenomenologically equivalent to point, and 4 macroscopic dimensions, phenomenologically equivalent to Galilean, Minkowski, de Sitter, anti de Sitter or some other space–times.

The model is theoretically relevant, because the space has sufficient room to provide the standard model symmetry $\mathbf{SU}(3) \times \mathbf{SU}(2) \times \mathbf{U}(1)$. However, as it is with all string theories, the space symmetry is much richer than physically observable. The symmetry can be shrinked by choosing $r_2, ..., r_{n-4} \neq 1$, still the symmetry group of this space is significally larger that necessary.

5 Further discussion

The essence of the Einstein equation in general relativity theory and its further development is to correlate the space–time curvature and the matter–energy interaction. Informally speaking, the presence of matter–energy defines the space–time curvature with its geometry, which in turn dictates how the matter has to move and energy has to propagate. However, the space–time curvature is exactly the characteristic r_{n-3} of proposed model. The equation doesn't include the rest of specification characteristics. More broader approach would be to write the equation that defines all n characteristics of the space specification, and thus, whole its geometry.

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Construction of pointwise lattices in Euclidean and Minkowski spaces with some given properties

Lilia Solovei

Abstract

In this paper concrete construction methods of the several Bravais types of pointwise lattices in Euclidean and Minkowski spaces depending on some additional conditions have been elaborated.

Keywords: pointwise lattice, Bravais type, Euclidean space, Minkowski space, crystallography.

1 Introduction

Let $\mathbb{R}_n$ (or ${}^1\mathbb{R}_n$) be the Euclidean n-dimensional pointwise space (or Minkowski space) and $\mathfrak{R}_n$ be a pointwise lattice in this space, i.e. $\mathfrak{R}_n$ is a set of points which have integer coordinates with respect to some fixed basis:

$$\mathfrak{R}_n = \{M : \overrightarrow{OM} = \sum_{i=1}^{n} x_i \vec{a}_i, x_i \in \mathbb{Z}\}.$$

The basis $\{O, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is the main basis of the lattice $\mathfrak{R}_n$. The properties of the lattices depend on their transformations of symmetry. A transformation of symmetry of the pointwise lattice is a motion of space that maps this lattice onto itself. The most simple transformations of symmetry of a lattice are the parallel translations by any of its vector. The set of all the parallel translations of each lattice is a commutative group generated by the parallel translations determined by the vectors of a certain main basis. Other transformations of the symmetry of the lattice either keep invariant at least one point (these transformations are called pointwise transformations), or are compositions of pointwise transformations with parallel translations. Rotations around k-dimensional planes k=$1, 2, \ldots, n-2$ as well as reflections through hyperplanes are transformations of symmetry. The pointwise transformations keeping invariant one common point also form a group named pointwise group of the lattice. The pointwise group in certain main basis of the lattice is determined by a group of integer unimodular matrices. The set of all the transformations of symmetry of the given lattice forms its complete group of symmetry. Two lattices are related to the same Bravais type if their complete groups of symmetry are isomorphic[1].

The main problem to be solved is to build all Bravais types of the lattices in the concrete given space. It is known that in Euclidean spaces the number of the Bravais types of pointwise lattices is finite and depends only on the dimension n (there are 5 types for n=2, 14 types for n=3, 64 types for n=4). In Minkowski spaces there exists an infinite number of Bravais types of pointwise lattices. Since the problem cannot be solved at all in many cases, it is necessary to elaborate certain methods of constructing Bravais types of pointwise lattices in the given space (Euclidean or Minkowski) that possess some of the earlier properties (for example, given the characteristics of the angle of rotation around a k-dimensional space, $k = 1, 2 \ldots, n-2$, either given the characteristics of the corresponding quadratic forms, or the lattice elements of symmetry are indicated etc.).

2 Construction of the pointwise lattices having a k-dimensional plan of rotation

Let us give further some methods of constructing the Bravais types of pointwise lattices in Euclidean and Minkowski spaces depending on some additional conditions. If the pointwise lattice $\mathfrak{R}_n$ in Euclidean or Minkowski space is mapped onto itself by a rotation of the space around a k-dimensional plane $\mathcal{G}$, $k = 1, 2, \ldots, n-2$, then it is mapped onto itself by a rotation around any k-dimensional plan which is parallel to $\mathcal{G}$ and is of the same nature, but passes through a certain point of the lattice. Let us suppose that the plane of rotation $\mathcal{G}$ passes through the point O of the lattice. We construct a $(n-k)$-dimensional plane $\mathcal{P}$ that passes through the point O and is absolutely perpendicular on $\mathcal{G}$. It is knows that each of these planes contains a k-dimensional sublattice ($(n-k)$-dimensional respectively)[2]. If the plane $\mathcal{G}$ is a Euclidean or Minkowski plane, then $\mathcal{G} \cap \mathcal{P} = \{O\}$ and the lattice consists of $(n-k)$-dimensional sublattices that are situated in planes parallel to $\mathcal{P}$ intersecting $\mathcal{G}$ in the points of the sublattices in this plane. These sublattices are called fibers. Let us enumerate these fibers by integers. We suppose that the fiber in the plane $\mathcal{P}$ is the null one; let us order points of the sublattice in the plane $\mathcal{G}$ and enumerate them by integers (obviously, the set of points of any lattice is countable). In Euclidean spaces all planes are also Euclidean, and if they are absolutely perpendicular, then their intersection is a single point ($\mathcal{P} + \mathcal{G} = \mathbb{R}_n$). In Minkowski spaces the plane $\mathcal{G}$ can be semi-Euclidean, then $\mathcal{P}$ is semi-Euclidean plane too. In such a case $\mathcal{G} \cap \mathcal{P} = \ell$, ℓ being an isotropic strainght line and $\mathcal{P} + \mathcal{G} = \mathbb{R}^{(1)}_{n-1}$, $\mathbb{R}^{(1)}_{n-1}$ being a semi-Euclidean hyperplane. In this case the hyperplane $\mathbb{R}^{(1)}_{n-1}$ contains an $(n-1)$-dimensional sublattice, and the lattice $\mathfrak{R}_n$ consists of the fibers parallel to one in this hyperplane. In order to construct the corresponding lattice it is sufficient to build the fiber in the hyperplane $\mathbb{R}^{(1)}_{n-1} = \mathcal{P} + \mathcal{G}$ and to multiply it by translations that are given by the vectors $S \cdot \vec{a}$, $S \in \mathbb{Z}$, $\vec{a} \parallel \mathbb{R}^{(1)}_{n-1}$.

It is obvious that the situation between any two neighbouring enumerated fibersis common for the whole lattice. Therefore, in order to construct all Bravais types of the pointwise lattices in any n-dimensional Euclidean or Minkowski space that are mapped onto themself by a rotation around a given k-dimensional plane $\mathcal{G}$ ($k = 1, 2, \ldots, n-2$), it is necessary: 1) to determine the dimension and the nature of space in which the lattices are built, as well as the dimension and the nature of the plane (or axis) of rotation; 2) to determine the nature of the planes absolutely perpendicular to the plan of rotation; 3) to establish the Bravais types of the pointwise lattices in $(n-k)$-dimensional spaces situated in a Euclidean or Minkowski space of the same dimension; 4) to examine the possibilities of intersection of the planes $\mathcal{P}$ and $\mathcal{G}$ ($\mathcal{G}$ is fixed plane of rotation); 5) to compare the built lattices and to establish the Bravais types of these pointwise lattices in the n-dimensional space. If the groups of matrices are integer equivalent, then the lattices are related to the same Bravais types.

3 Conclusion

Based on the properties of the elements of symmetry of the pointwise lattices which are the intersection of the other elements of symmetry some methods of construction of the Brawais types of pointwise lattices from Euclidean and Minkovski n-dimensional space have been developed. The finding may be useful in construction of the lattices having a k-dimensional plane of rotation.

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On isohedral tilings for hyperbolic group of genus 2 by tiles with large number of edges

Elizaveta Zamorzaeva

Abstract

The paper studies isohedral tilings of the hyperbolic plane for the group of translations of genus 2 by 14-gons. It is shown how to enumerate all eligible cycles of valencies for polygon with large number of vertices.

Keywords: isohedral tilings, hyperbolic plane, group of translations, genus two.

On the Euclidean plane there are 2 types as well as 2 Delone classes of isohedral tilings by disks for the group of translations $p1$. Those disks are parallelograms and center-symmetric hexagons (if convex). On the hyperbolic plane there are a countable series of discrete groups of translations with compact fundamental domain, each group of the series is characterized by its genus. The simplest hyperbolic group of translations has genus 2, its Conway's orbifold symbol is $\circ\circ$. So I study isohedral tilings for the hyperbolic translation group of genus 2.

Definition. *Let W be a tiling of the hyperbolic plane with disks, G be a discrete isometry group with a bounded fundamental domain. The tiling W is called isohedral with respect to the group G if G acts transitively on the set of all disks of the tiling.*

Isohedral tilings of the hyperbolic plane are enumerated up to Delone classes. A Delone class takes in consideration a tiling W, a group G and the action of the group G on the tiles of W (see [1]).

Any translation group admits only fundamental Delone classes of tilings, i. e. tilings of the plane with fundamental domains. To find isohedral tilings for the hyperbolic translation group of genus 2, I have solved some Diophantine equations in integers, then use B. Delone's adjacency diagrams and adjacency symbols [2] (see also [1]).

In the present communication I would like to dwell upon some new difficulties that arise when dealing with polygons with a large number of vertices.

The solutions to Diophantine equatons give us $k = 8$, 10, 12, 14, 16, and 18 as the numbers of vertices of a polygon. For $k = 14$ there are two sets of vertex valencies: the set A contains nine 3 and five 5, the set B contains six 3 and eight 4. From the sets of valencies one should form all possible ordered cycles of valencies up to equivalence.

Choose the set B : 33333344444444 and proceed to form possible classes of ordered cycles. In order to avoid superfluous examination we take in consideration that edges of a 14-gon must fall into pairs. The edges of a pair are mapped one into another by translation, so in a pair two edges have ends of the same valencies, in appropriate order. Thus we enumerate only eligible cycles of valencies.

For valencies 3 take into account one more forbidden situation: neighboring vertices of valency 3 cannot be sent one into another. Now examine possible groups of consecutive numbers of valencies from six 3 and eight 4. In an eligible cycle the group 333333 is not admissible because of yielding 5 edges 33 (both ends have valency 3). The arrangement of 33333 and a separate 3 as well as the arrangement of 3333 and 33 (both giving 4 edges 33) is not admissible because of the above mentioned forbidden situation. The arragement of 3333, separate 3, and 3 is not admissible because of yielding 3 edges 33. The arrangement of 333 and 333 yields 4 edges 33 and leads to some eligible ordered cycles of valencies:

$B1$: 33343334444444, $B2$: 33344333444444, $B3$: 33344433344444, $B4$: 33344443334444.

The arrangement of 333, 33, and separate 3 yields 3 edges 33 and is not admissible. The arrangement of 333, seperate 3, 3, and 3 yields 2 neighboring edges 33 and is not admissible. The arrangement of 33, 33, separate 3, and 3 yields 2 edges 33 and can be used in different combinations with eight vertices of valency 4. Now examine different

ways to divide 44444444 into four groups. First verify that the arrangement of 44444, separate 4, 4, and 4 is not admissible because, for 4 consecutive edges 44, no pairing allows to go round a vertex of valency 4. The arrangement of 4444, 44, separate 4, and 4 yields 4 edges 44 and leads to the following eligible ordered cycles of valencies:
$B5$: 33433434434444, $B6$: 33433443434444, $B7$: 33434334434444, $B8$: 33434433434444, $B9$: 33443343434444, $B10$: 33443433434444, $B11$: 33443343444434, $B12$: 33443433444434, $B13$: 33434344334444, $B14$: 33434434334444, $B15$: 33443434334444, $B16$: 33443434444334, $B17$: 33433434444344, $B18$: 33434334444344, $B19$: 33434344443344, $B20$: 33433443444434, $B21$: 33434433444434, $B22$: 33434434444334, $B23$: 33433444434344, $B24$: 33434444334344, $B25$: 33434444343344, $B26$: 33433444434434, $B27$: 33434444334434, $B28$: 33434444344334, $B29$: 33443344443434, $B30$: 33443444433434, $B31$: 33443444434334, $B32$: 33444433434344, $B33$: 33444434334344, $B34$: 33444434343344, $B35$: 33444433434434, $B36$: 33444434334434, $B37$: 33444434344334, $B38$: 33444433443434, $B39$: 33444434433434, $B40$: 33444434434334.

The combinations of the arrangement of 444, 444, separate 4, and 4 with the arrangement of 33, 33, separate 3, and 3 yielding 2 edges 33, 8 edges 34 and 4 edges 44 lead to the following eligible ordered cycles of valencies:
$B41$: 33433434443444, $B42$: 33434334443444, $B43$: 33434344433444, $B44$: 33433444343444, $B45$: 33434443343444, $B46$: 33434443433444, $B47$: 33433444344434, $B48$: 33434443344434, $B49$: 33434443444334, $B50$: 33444334343444, $B51$: 33444343343444, $B52$: 33444343433444, $B53$: 33444334344434, $B54$: 33444343344434, $B55$: 33444343444334, $B56$: 33444334443434, $B57$: 33444344433434, $B58$: 33444344434334.

The combinations of the arrangement of 44, 44, 44, and 44 with the arrangement of 33, 33, separate 3, and 3 leads to the following eligible ordered cycles of valencies:
$B59$: 33443344344344, $B60$: 33443443344344.

If all six vertices of valency 3 are separate, then eight vertices of valency 4 should be divided into six groups. The arrangement of 44, 44, and all other separate 4 is not admissible because it yields 2 consecutive

edges 44. So the arrangement of 44, 44, and all other separate 4 is the only admissible in this case, and it leads to the following eligible ordered cycles of valencies:
$B61 : 34343434344344$, $B62 : 34343434434344$, $B63 : 34343443434344$.

Thus all classes of eligible cycles of valencies are enumerated for the set $14B$. For each class of ordered cycles of valencies, we form all possible adjacency diagrams. Then to every adjacency diagram we apply the procedure of going round all the vertices of a tile and verify whether the condition is satisfied. Altogether for the list of $14B$, there are 36 Delone classes of isohedral tilings. Each Delone class is given by an adjacency symbol.

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Section 3

Analysis, Differential Equations and Dynamical Systems

Averaging in Multifrequency Systems with Linearly Transformed Arguments and with Point and Integral Conditions

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Abstract

The multifrequency system of equations with linearly transformed arguments and with point and integral conditions is considered. The existence and uniqueness of solution of the problem are investigated. The averaging method is justified based on evaluation of oscillating integrals and the estimation error of averaging method for slow variables is obtained.

Keywords: averaging method, multifrequency systems, resonance, linearly transformed argument.

1 The scheme of averaging

Multifrequency systems of ordinary differential equations with initial and integral conditions were investigated in [1] with averaging method. Similar problems for differential equations with delay and linearly transformed arguments were explored in [2-4]. Averaged system is more simple, because equation for slow variables a does not depend on fast variables φ.

We consider a system of differential equations

$$\frac{da}{d\tau} = X(\tau, a_\Lambda, \varphi_\Theta), \qquad \frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau, a_\Lambda, \varphi_\Theta), \tag{1}$$

where $\tau \in [0, L]$, $a \in \mathbb{D} \subset \mathbb{R}^n$, $\varphi \in \mathbb{T}^m$, $m \geqslant 1$, $0 < \varepsilon \leqslant \varepsilon_0 \ll 1$, $\Lambda = (\lambda_1, \ldots, \lambda_r)$, $\Theta = (\theta_1, \ldots, \theta_s)$, $\lambda_i, \theta_j \in (0;1]$, $a_\Lambda = (a_{\lambda_1}, ..., a_{\lambda_r})$, $a_{\lambda_i}(\tau) = a(\lambda_i \tau)$, $i = \overline{1, r}$, φ_Θ=$(\varphi_{\theta_1}, \ldots, \varphi_{\theta_s})$, $\varphi_{\theta_j}(\tau) = \varphi(\theta_j \tau)$, $j = \overline{1, s}$.

Let us set next conditions for the solutions of system (1)

$$a(\tau_0)=a_0,\quad \int\limits_{\tau_1}^{\tau_2}\Big[\sum_{j=1}^{s} b_j(\tau,a_\Lambda(\tau))\varphi_{\theta_j}(\tau)+g(\tau,a_\Lambda(\tau),\varphi_\Theta(\tau))\Big]d\tau=d, \quad (2)$$

where $0 \leqslant \tau_0 \leqslant L, 0 \leqslant \tau_1 < \tau_2 \leqslant L$.

Averaged for all fast variables φ_{θ_j} system takes form

$$\frac{d\overline{a}}{d\tau} = X_0(\tau,\overline{a}_\Lambda), \qquad \overline{a}(\tau_0) = a_0, \quad (3)$$

$$\frac{d\overline{\varphi}}{d\tau}=\frac{\omega(\tau)}{\varepsilon}+Y_0(\tau,\overline{a}_\Lambda),\quad \int\limits_{\tau_1}^{\tau_2}\Big[\sum_{j=1}^{s} b_j(\tau,\overline{a}_\Lambda(\tau))\overline{\varphi}_{\theta_j}(\tau)+g_0(\tau,\overline{a}_\Lambda(\tau))\Big]d\tau=d. \quad (4)$$

Let $S_R = \{a : \|a-a_0\| \leqslant R\}$, $\max\limits_{[0,L]\times S_R^r} \|X_0(\tau,a_\Lambda)\| \leqslant \sigma$.

Theorem 1. *Let us suppose that function $X(\tau,\overline{a}_\Lambda)$ is continuous with set of variables in the area $[0,L]\times S_R^r$, satisfies Lipschitz condition for variable a_Λ with constant $\alpha > 0$ and $\sigma L \leqslant R$, $\alpha r L < 1$. Then solution of the problem (3), (4) exists and is unique.*

2 Conditions and justification of the averaging method

Let $G := [0,L]\times D^r$, $G_1 := G\times T^{ms}$, $f := (X,Y,g)$. Let us suppose that

1^0. $f \in \mathbb{C}^2_{a_\Lambda}(G_1,\sigma)$, $f \in \mathbb{C}^1_\tau(G_1,\sigma)$, where norms of function f and its derivatives is limited with σ.

2^0. $f \in \mathbb{C}^{mr+1}_{\varphi_\Theta}(G_1,\sigma)$.

3^0. $b_i \in \mathbb{C}^2(G,\beta_i)$, $j=\overline{1,s}$.

4^0. $\omega \in \mathbb{C}^{ms-1}([0,L],\sigma)$ and Wronskian $V(\tau)$, built with system of functions $\big\{\omega(\theta_1\tau),\ldots,\omega(\theta_s\tau)\big\}$ is not 0 for $\tau\in[0,L]$.

5^0. There is unique solution of the problem (3), (4) that is lying in area D with some ρ-neighborhood.

6^0. Matrixes

$$I-\sum_{j=1}^{s}\int_{0}^{\tau_0}\frac{\partial X_0(\tau,\overline{a}_\Lambda(\tau,\overline{y}))}{\partial a_{\lambda_j}}\frac{\partial a_{\lambda_j}(\tau,\overline{y})}{\partial\overline{y}}d\tau \quad\text{and}\quad \sum_{j=1}^{s}\int_{\tau_1}^{\tau_2}b_j(\tau,\overline{a}_\Lambda(\tau))d\tau$$

are invertible.

Condition 3^0 provides that solution of system (3) goes through small neighborhood of the resonance. The condition of the resonance for $\tau\in[0,L]$ have the form [2, 3] $\sum\limits_{j=1}^{s}\theta_j(k_j,\omega(\theta_j\tau))=0$, $k_j\in\mathbb{Z}^m$, $\sum\limits_{j=1}^{s}\|k_j\|\neq 0$.

Theorem 2. *Let us suppose that conditions 1^0–6^0 are satisfied. Then for small enough ε the unique solution of problem (1), (2) exists and for all $\tau\in[0,L]$ and $\varepsilon\in(0,\varepsilon_0]$ performs evaluation*

$$\|a(\tau,\overline{y}+\mu(\varepsilon),\overline{\psi}+\xi(\mu(\varepsilon),\varepsilon),\varepsilon)-\overline{a}(\tau,\overline{y})\|+$$

$$\|\varphi(\tau,\overline{y}+\mu(\varepsilon),\overline{\psi}+\xi(\mu(\varepsilon),\varepsilon),\varepsilon)-\overline{\varphi}(\tau,\overline{y},\overline{\psi},\varepsilon)-\eta(\varepsilon)\|\leqslant c_1\varepsilon^\alpha,$$

where $\alpha=(rm)^{-1}$, $\eta\in\mathbb{R}^m$ and $\|\eta(\varepsilon)\|\leqslant c_2\varepsilon^{\alpha-1}$.

The evaluation for the intergal

$$I_k(\tau,\varepsilon)=\int_0^\tau f(z,\varepsilon)\exp\left(\frac{i}{\varepsilon}\int_0^z\gamma_k(t)dt\right)dz$$

that corresponds system (1) was used for theorem proving.

In [2] there was shown that

$$\|I_k(\tau,\varepsilon)\|\leqslant c_3\left(\sup\|f(t,\varepsilon)\|+\frac{1}{\|k\|}\sup\left\|\frac{df}{dt}\right\|\right),$$

where $c_3>0$ and does not depend on ε and k.

Remark 1. The result of theorem 1 is correct also for multifrequency system (1), (2) with linearly transformed arguments $\lambda_i:[0,L]\to[0,L]$, $\theta_j:[0,L]\to[0,L]$ where $\lambda_i'(\tau)\neq 0$ and $\theta_j'(\tau)\neq 0$ for $\tau\in[0,L]$, $i=\overline{1,r}$, $j=\overline{1,s}$. The condition 4^0 is formulating for the determinant $V(\tau)$ with the vectors $\omega^T(\tau), \omega^T(\theta_1(\tau))\theta_1'(\tau), \ldots,$ $\omega^T(\theta_s(\tau))\theta_s'(\tau)$ in first row, T means transposition. For instance, for $m = s = 1$ condition 4^0 takes the form

$$det \begin{bmatrix} \omega(\tau) & \omega(\theta(\tau))\theta'(\tau) \\ \omega'(\tau) & \omega'(\theta(\tau))\big(\theta'(\tau)\big)^2 + \omega(\theta(\tau))\theta''(\tau) \end{bmatrix} \neq 0, \qquad \tau \in [0, L].$$

Remark 2. If condition 4^0 is not satisfied in points τ_ν, $\nu = \overline{1, \rho}$ and the multiplicity of roots of the equation $V(\tau) = 0$ is limited by the number $q \geq 1$, then theorem 1 can be proved also in this case.

Let us apply the proving scheme suggested in [1] for ordinary differential equations and in [2] for system of differential equations with linearly transformed arguments. Herewith the error of averaging method for slow variables takes the form $\|a(\tau, \varepsilon) - \overline{a}(\tau)\| \leq c\varepsilon^\beta$, where $\beta = (rm + q)^{-1}$, $c = const > 0$.

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Phase portraits of cubic differential systems with invariant straight lines of total multiplicity eight

Cristina Bujac, Nicolae Vulpe

Abstract

In this article for the family of cubic differential systems with eight invariant straight lines considered with their multiplicities all the phase portraits were constructed. For such systems the classification according to the configurations of invariant lines in terms of affine invariant polynomials were done in [1–5] and all possible 51 configurations were constructed. For each one of the 51 such classes we perform its corresponding phase portraits and prove that only 30 such phase portraits are topologically distinct.

Keywords: Cubic differential system, phase portrait, configuration of invariant lines, group action, affine invariant polynomial.

1 Introduction

In the article [6] for the family of cubic differential systems with the maximum number of invariant straight lines, i.e. 9 (considered with their multiplicities), all the phase portraits were constructed. Our paper is a continuation of [6] and namely, here we consider the phase portraits of the class $\mathbb{CSL}_8$ of cubic systems possessing invariant lines of total multiplicity 8 (including the line at infinity).

Our work is based on the results obtained in [1–5], where for systems in $\mathbb{CSL}_8$ the classification according to the configurations of invariant lines in terms of affine invariant polynomials was done. As a result there was proved the existence of exactly 51 distinct configurations of invariant lines as well as the necessary and sufficient conditions of the realization of each one of them.

2 Main results

Main Theorem. *Consider a non-degenerate cubic system and assume that it belongs to the family* $\mathbb{CSL}_8$, *i.e. it possesses one of the 51 possible configurations of invariant lines Config. 8.j* ($j = 1, \ldots, 51$) *detected in [1–5]. Then:*

(*A*) *the phase portrait of this system correspond to one of the 52 phase portraits P. 8.1–P. 8.5, P. 8.6(a), P. 8.6(b), P. 8.7–P. 8.51 given in Figure 2;*

(*B*) *among 52 phase portraits given in Figure 2 there are exactly 30 topologically distinct phase portraits as it is indicated in Diagram 2 using the geometric invariants defined in Remark 1. Moreover applying the algebraic theory of invariants the necessary and sufficient conditions for the realization of each one of the detected 30 topologically distinct phase portraits where established.*

Remark 1. *In order to distinguish topologically the phase portraits of the systems we obtained, we use the following geometric invariants:*

- *The number* $IS^{\mathbb{R}}$ *of real infinite singularities.*
- *The number* $FS^{\mathbb{R}}$ *of real finite singularities.*
- *The number* Sep^f *of separatrices associated to finite singularities.*
- *The number* Sep^∞ *of separatrices associated to infinite singularities.*
- *The number* $FSep$ *of separatrices connecting finite singularities.*
- *The number* SC *of separatrix connections.*
- *The maximum number* ES^∞ *of elliptic sectors in the vicinity of an infinite singularity.*

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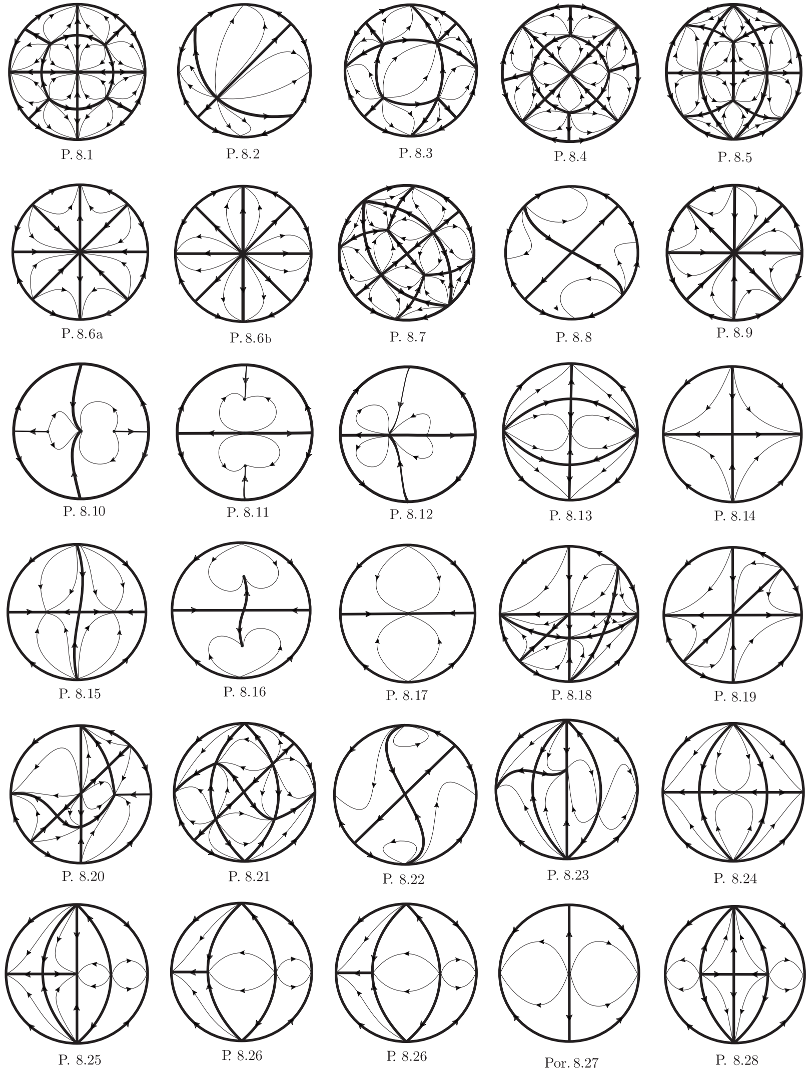

Figure 1. Phase portraits of systems in $\mathbb{CSL}_8$.

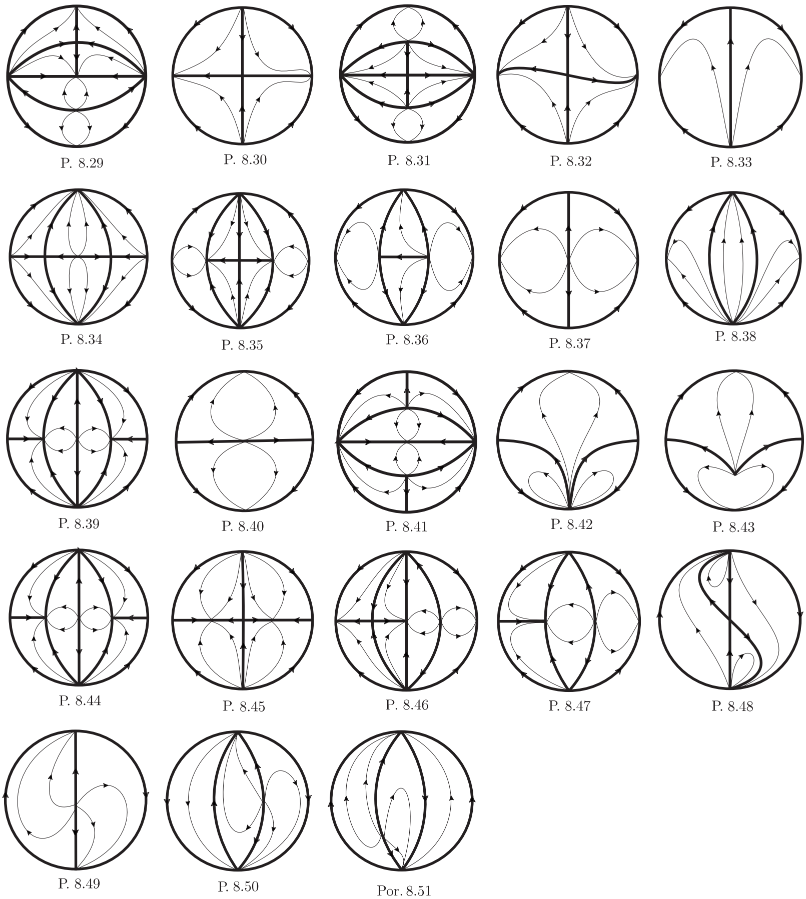

Figure 2 *(continuation)*. Phase portraits of systems in $\mathbb{CSL}_8$.

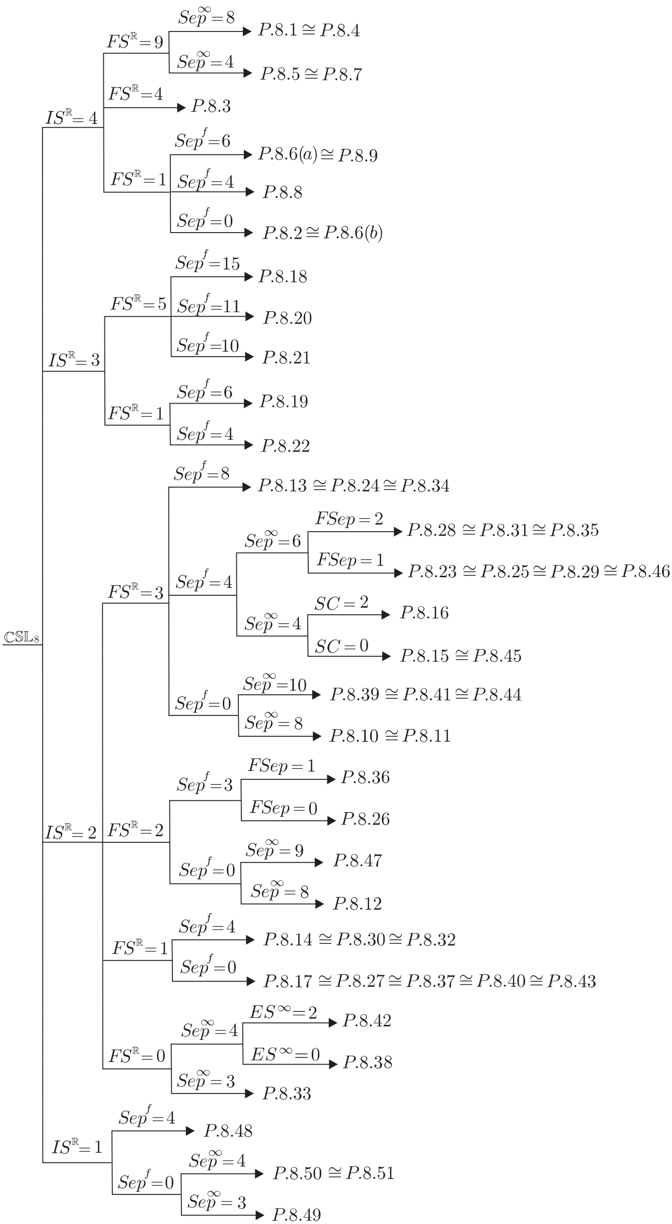

Diagram 1. Topologically distinct phase portraits

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Up continuity

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Abstract

In this paper, we introduce and investigate the concept of *up continuity*. It turns out that the set of up continuous functions is a proper subset of the set of continuous functions.
Keywords: Sequences, series, summability, continuity.

1 Introduction

Using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([7], [15]), quasi-slowly oscillating continuity ([9]), ward continuity (, [1], [8]), strongly lacunary ward continuity ([5], [10]), which enabled some authors to obtain conditions on the domain of a function to be uniformly continuous in terms of sequences in the sense that a function preserves a certain kind of sequences (see for example [15, Theorem 6],[1, Theorem 1 and Theorem 2],[9, Theorem 2.3].

The purpose of this paper is to introduce the concepts of up compactness of a subset of the set of real numbers and up continuity of a real function, which cannot be given by means of a sequential method G and prove interesting theorems.

2 Up continuity

A sequence (α_k) in $\mathbf{R}$ is called *upward half Cauchy* if for each $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ so that $\alpha_n - \alpha_m < \varepsilon$ for $m \geq n \geq n_0$ ([12]).

Definition 1. A subset E of $\mathbf{R}$ is called *up compact*, if any sequence of points in E has an upward half Cauchy subsequence.

Theorem 1. A subset of $\mathbf{R}$ is up compact if and only if it is bounded below.

It follows from the above theorem that if a closed subset A of $\mathbf{R}$ is up compact, and $-A$ is up compact, then any sequence of points in E has a (P_n, s)-absolutely almost convergent subsequence (see [4], [16], and [17]).

Known results for continuity for real functions in terms of sequences might suggest to us introducing a new type of continuity, namely, up continuity.

Definition 2. A function $f : E \to \mathbb{R}$ is called *up continuous* on a subset of $\mathbf{R}$, if it preserves upward half Cauchy sequences, i.e. the sequence $(f(\alpha_n))$ is upward half Cauchy whenever (α_n) is an upward half Cauchy sequence of points in E.

We see that the sum of two up continuous functions is up continuous.

Theorem 2. Any up continuous function is continuous.

Theorem 3. Up continuous image of any up half compact subset of $\mathbf{R}$ is up half compact.

Theorem 4. If (f_n) is a sequence of up continuous functions defined on a subset E of $\mathbf{R}$ and (f_n) is uniformly convergent to a function f, then f is up continuous on E.

3 Conclusion

In this paper, we have obtained results related to up compactness, and up continuity, and some other kinds of continuities via upward half Cauchy sequences, convergent sequences, statistical convergent sequences, lacunary statistical convergent sequences of points in $\mathbf{R}$. It turns out that the set of up continuous functions is a proper subset of the set of ordinary continuous functions. The term upward half Cauchy sequence can be considered to be associated with below boundedness of the underlying space, whereas the term Cauchy sequence is traditionally associated with the completeness of the underlying space. We suggest to investigate upward half Cauchy sequences of fuzzy points in asymmetric fuzzy spaces (see [11], for the definitions and related concepts in fuzzy setting). We also suggest to investigate upward half Cauchy double sequences (see for example [13] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate upward half Cauchy sequences of points in an asymmetric cone metric space since in a cone metric space the notion of an upward half Cauchy sequence coincides with the notion of a Cauchy sequence, and therefore up continuity coincides with continuity (see [14]).

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Linear Stochastic Differential Equations and Nonautonomous Dynamical Systems

David Cheban

Abstract

We prove that the linear stochastic equation $dx(t) = (Ax(t) + f(t))dt + g(t)dW(t)$ (*) with linear operator A generating a C_0-semigroup $\{U(t)\}_{t\geq 0}$ and Levitan almost periodic forcing terms f and g admits a unique Levitan almost periodic [3,ChIV] solution in distrution sense if it has at least one precompact solution on $\mathbb{R}_+$ and the semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically stable.

Keywords: Levitan almost periodic solutions, linear stochastic differential equations.

1 Introduction

In this short communication we study the problem of existence of Levitan almost periodic solutions of equations (*), where A is generator of strongly asymptotically stable C_0-semigroup on a Banach space E and $f, g :\to E$ are some Levitan almost periodic functions.

In the deterministic case ($g = 0$) the problem of Bohr almost periodicity (respectively, almost automorphy) of solutions of equation (*) was studied in the works of S. Zaidman [2] (for Bohr almost periodic equations) and M. Zaki [3] (for almost automorphic equations).

2 Semigroup of operators

Let $(E, |\cdot|)$ be a Banach space with the norm $|\cdot|$ and $[E]$ be a Banach space of linear bounded operators A acting on the space A equipped with the norm $||A|| := \sup\{|Ax| : \ |x| \leq 1\}$.

A C_0-semigroup $\{U(t)\}_{t\geq 0}$ is said to be asymptotically stable if $\lim\limits_{t\to+\infty} U(t)x = 0$ for any $x \in E$.

Theorem 1.[1,ChI] *The following statements are equivalent:*

1. *the C_0-semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically stable;*
2. $\lim\limits_{t\to+\infty} \sup\limits_{x\in K} |U(t)x| = 0$ *for any compact subset $K \subset E$;*
3. *equation $x'(t) = Ax(t)$*
   - (a) *admits a compact global attractor J;*
   - (b) *does not admit any solution defined on $\mathbb{R}$ with precompact range, i.e., $J = \{0\}$.*

Let (X, ρ) be a compete metric space. Denote by $C(\mathbb{R}, X)$ the family of all continuous functions $f : \mathbb{R} \mapsto X$ equipped with the distance $d(f, g) := \sup\limits_{l>0} d_l(f, g)$, where $d_l(f, g) := \min\{\max\limits_{|t|\leq l} \rho(f(t), g(t)); l^{-1}\}$. The metric d is complete and it defines on $C(\mathbb{R}, X)$ the compact-open topology. Let $h \in \mathbb{R}$ denote by f_h the h-translation of f, that is, $f_h(s) := f(s + h)$ for all $s \in \mathbb{R}$.

Definition 1. *A function $f \in C(\mathbb{R}, X)$ is said to be Bohr almost periodic if for any $\varepsilon > 0$ there exists a positive number $L = L(\varepsilon)$ such that $\mathcal{T}(\varepsilon, f) \bigcap [a, a + L] \neq \emptyset$ for any $a \in \mathbb{R}$, where $\mathcal{T}(\varepsilon, f) := \{\tau \in \mathbb{R} : \rho(f(t + \tau), f((t))) < \varepsilon$ for any $t \in \mathbb{R}\}$.*

Definition 2. *Function $f \in C(\mathbb{R}, X)$ is called Levitan almost periodic if there exists a metric space Y and a Bohr almost periodic function $F \in C(\mathbb{R}, Y)$ such that for arbitrary $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $\mathcal{T}(\delta, F) \subseteq \mathfrak{T}(\varepsilon, f)$, where $\mathfrak{T}(\varepsilon, f) := \{\tau \in \mathbb{R} : \max\limits_{|t|\leq 1/\varepsilon} \rho(f(t + \tau), f(t)) < \varepsilon\}$.*

Remark 1. 1.*Every Bohr almost periodic function is Levitan almost periodic.*

2.*The functions $f(t) = (2 + \cos t + \cos \sqrt{2}t)^{-1}$ and $g(t) = \cos(f(t))$ ($t \in \mathbb{R}$) are Levitan almost periodic, but not Bohr almost periodic [3,ChIV].*

3 Linear Stochastic Differential Equations

Let $(H, |\cdot|)$ be a real separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $L^2(\mathbb{P}, H)$ be the space of H-valued random variables x such that $\mathbb{E}|x|^2 := \int\limits_\Omega |x|^2 d\mathbb{P} < \infty$. Then $L^2(\mathbb{P}, H)$ is a Hilbert space equipped with the norm $||x||_2 := \Big(\int\limits_\Omega |x|^2 d\mathbb{P}\Big)^{1/2}$.

Consider the following linear stochastic differential equation

$$dx(t) = (Ax(t) + f(t)dt + g(t)dW(t), \tag{1}$$

where A is an infinitesimal generator which generates a C_0-semigroup $\{U(t)\}_{t\geq 0}$, $f, g \in C(\mathbb{R}, H)$ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We set $\mathcal{F}_t := \sigma\{W(u) : u \leq t\}$.

Recall that an $\mathcal{F}_t$-adapted processes $\{x(t)\}_{t\in\mathbb{R}}$ is said to be a mild solution of equation (1) if it satisfies the stochastic integral equation

$$x(t) = U(t - t_0)x(t_0) + \int_{t_0}^{t} U(t - s)f(s)ds + \int_{t_0}^{t} U(t - s)g(s)dW(s),$$

for all $t \geq t_0$ and each $t_0 \in \mathbb{R}$.

Let $\mathcal{P}(H)$ be the space of all Borel probability measures on H endowed with the weak topology. It is well known that on the space $\mathcal{P}(H)$ there is a distance which defines this topology.

Definition 3. *Let $\varphi : \mathbb{R} \to E$ be a mild solution of equation* (1)*. Then φ is called Levitan almost periodic* in distribution *if the function $\phi \in C(\mathbb{R}, \mathcal{P}(H))$ is Levitan almost periodic, where $\phi(t) := \mathcal{L}(\varphi(t))$ for any $t \in \mathbb{R}$ and $\mathcal{L}(\varphi(t)) \in \mathcal{P}(H)$ is the law of random variable $\varphi(t)$.*

Theorem 2. *Suppose that the following conditions are fulfilled:*

a. *the C_0-semigroup $\{U(t)\}_{t\geq 0}$ is asymptotically stable;*

b. *the functions $f, g \in C(\mathbb{R}, H)$ are Levitan almost periodic;*

c. *equation (1) admits a solution φ defined on $\mathbb{R}_+$ with precompact range, i.e., the set $Q := \overline{\varphi(\mathbb{R}_+)}$ is compact.*

Then equation (1) has a unique solution p defined on $\mathbb{R}$ with precompact range which is Levitan almost periodic in distribution sense and $\lim\limits_{t\to+\infty} |\varphi(t) - p(t)| = 0.$

To prove this statement we use some ideas, methods and results from the theory of nonautonomous (cocycle) dynamical systems [1].

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Decomposition of multiparameter linear singularly perturbed systems

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Abstract

We investigate a system of linear singularly perturbed differential equations with plenty of small parameters. The algorithm of the decomposition scheme contains k steps of a successive splitting based on the integral manifold of quick and slow variables method. The splitting substitution is defined constructively in the form of expansions by powers of small parameters.

Keywords: linear singularly perturbed systems, decomposition, asymptotic decomposition, integral manifolds.

1 Introduction

Constructive methods of decomposition of singularly perturbed systems, which are based on the ideas of the integral manifold method, were developed in the works [1-2]. These methods are effective only in cases where we can find precisely or approximately integral manifold. For singularly perturbed systems we can build integral manifolds in the form of expansion by powers of small parameters [3-4]. For linear singularly perturbed systems method of integral manifolds allows us to perform the splitting transformation of input system to independent fast and slow subsystems [5]. In this study we consider linear singularly perturbed systems with several small parameters [6].

2 Main results

Consider a linear singularly perturbed system given by

$$\prod_{j=0}^{i} \varepsilon_j \dot{x}_i = \sum_{j=0}^{k} A_{ij} x_j, \; i = \overline{0,k}, \tag{1}$$

where $t \in \mathbb{R}$, $x_i \in \mathbb{R}^{n_i}$, $A_{ij} = A_{ij}(t)$, $i, j = \overline{0,k}$, are $n_i \times n_j$ matrices, $\varepsilon_0 = 1$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ are small positive parameters.

Let the following conditions be true:

1) matrices $A_{ij}(t)$, $i, j = \overline{0,k}$, are uniformly bounded in $t \in \mathbb{R}$,

2) eigenvalues of $\lambda_j = \lambda_j(t)$, $j = \overline{1,n_k}$, of the matrix $A_{kk}(t)$ satisfy the inequality

$$\mathrm{Re}\lambda_j(A_{kk}) \leq -2\beta < 0.$$

The decomposition of system (1) will be performed in k steps. At each step $(l+1), l = \overline{0, k-1}$, the substitution is made as follows

$$\begin{cases} y_i^l = y_i^{l+1} + \prod\limits_{j=i+1}^{k-l} \varepsilon_j H_i^{l+1} y_{k-l}^{l+1}, i = \overline{0, k-l}, \\ y_{k-l}^l = \sum\limits_{i=0}^{k-l-1} P_i^{l+1} y_i^l + y_{k-l}^{l+1}, \end{cases} \tag{2}$$

where $y_i^0 = x_i$.

We obtain the block-diagonal system

$$\begin{cases} \dot{y}_0^k = B_{00}^k y_0^k, \\ \prod\limits_{j=0}^{i} \varepsilon_j \dot{y}_i^{k-i+1} = B_{ii}^{k-i+1} y_i^{k-i+1}, i = \overline{1,k}, \end{cases} \tag{3}$$

Theorem 1. Let conditions 1)-2) be true. Then for sufficiently small parameters $\varepsilon_i, i = \overline{1,k}$, there exists a nonsingular substitution

$$(x_0, x_1, x_2, \ldots, x_k)^T = \Phi\left(y_0^k, y_1^k, y_2^{k-1}, \ldots, y_k^1\right)^T,$$

which transforms system (1) to $(k+1)$ independent subsystems (3).

Theorem 2 [8]. Let the conditions 1), 2) be true and let matrices $B^l_{ij}(i,j=\overline{0,k-l})$, $(B^l_{k-l,k-l})^{-1}$ and their $(n+1)$ derivatives be uniformly bounded in $t\in\mathbb{R}$. Then for sufficiently small ε_l there exists a substitution (2), which transforms system $\prod\limits_{j=0}^{i}\varepsilon_j\dot{y}^l_i=\sum\limits_{j=0}^{k-l}B^l_{ij}y^l_j$, $i=\overline{0,k-l}$, to the form

$$\begin{cases}\prod\limits_{j=0}^{i}\varepsilon_j\dot{y}^{l+1}_i=\sum\limits_{j=0}^{k-l-1}B^{l+1}_{ij}y^{l+1}_j, \quad i=\overline{0,k-l-1},\\ \prod\limits_{j=0}^{k-l}\varepsilon_j\dot{y}^{l+1}_{k-l}=B^{l+1}_{k-l,k-l}y^{l+1}_{k-l},\end{cases}\tag{4}$$

and the coefficients of the asymptotic decomposition of the transformation can be uniquely found by

$$\begin{aligned}&P^{l+1}_{i,0}(t)=-(B^l_{k-l,k-l})^{-1}(t)B^l_{k-l,i}(t),\\&P^{l+1}_{i,j}(t)=(B^l_{k-l,k-l})^{-1}(t)\left(\prod_{m=1}^{k-l-1}\varepsilon_m\dot{P}^{l+1}_{i,j-1}(t)+\right.\\&\quad+\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_s\sum_{s=0}^{j-1}P^{l+1}_{m,s}B^l_{m,k-l}P^{l+1}_{i,j-s-1}+\\&\quad\left.+\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_sP^{l+1}_{m,j-1}B^l_{mi}\right),\quad j=\overline{1,n},\end{aligned}\tag{5}$$

and

$$\begin{aligned}&H^{l+1}_{i,0}=B_{i,k-l}(B^l_{k-l,k-l})^{-1},\\&H^{l+1}_{i,j}=\left(\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-2}\varepsilon_sB^l_{im}H^{l+1}_{m,j-1}+\right.\\&\quad+\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_s\sum_{s=0}^{j-1}H^{l+1}_{i,s}P^{l+1}_{m,j-s-l}B^l_{m,k-l}+\\&\quad+\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_sB^l_{i,k-l}\sum_{s=0}^{j-1}P^{l+1}_{m,s}H^{l+1}_{m,j-1-s}-\\&\quad\left.-\prod_{m=0}^{k-l-1}\varepsilon_m\dot{H}^{l+1}_{i,j-1}\right)(B^l_{k-l,k-l})^{-1},\quad j=\overline{1,n}.\end{aligned}\tag{6}$$

3 Conclusion

By completing k steps of the decomposition of system (1) by the scheme described above, we obtain block-diagonal system (3). Moreover, the coefficients of the asymptotic decomposition of the transformation can be found by recurrent algebraic relations analogous to (5), (6).

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The Lyapunov quantities and the $GL(2,\mathbb{R})$-invariant center conditions for a class of bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree

Stanislav Ciubotaru, Iurie Calin

Abstract

For the bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree the recurent equations for determination of the Lyapunov quantities were established. Moreover, the general form of Lyapunov quantities for the mentioned systems were obtained. For a class of such systems the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for the existence of center are given.

Keywords: Polynomial differential systems, invariant, comitant, transvectant, Lyapunov quantities, center conditions.

1 Definitions and notations

Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = P_1(x,y) + P_4(x,y), \quad \frac{dy}{dt} = Q_1(x,y) + Q_4(x,y), \tag{1}$$

where $P_i(x,y),\ Q_i(x,y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

The goal of this paper is to determine the invariant [1, 2] recurent formulas for construction the Lyapunov quantities for the system (1) and to establish the invariant center conditions for a class of these systems.

Definition. [3] *Let φ and ψ be homogeneous polynomials in coordinates of the vector $(x,\ y) \in \mathbb{R}^2$ of the degrees ρ_1 and ρ_2, respectively. The polynomial*

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1-j)!(\rho_2-j)!}{\rho_1!\rho_2!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{\partial^j\varphi}{\partial x^{j-i}\partial y^i}\frac{\partial^j\psi}{\partial x^i\partial y^{j-i}}$$

is called the transvectant of index j of polynomials φ and ψ.

If the polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants [1, 2] of the system (1), then the transvectant of the index $j \le \min(\rho_1,\rho_2)$ is also a $GL(2,\mathbb{R})$-comitant of the system (1) [4].

$GL(2,\mathbb{R})$-comitants of the first degree with respect to coefficients of system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ \ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ \ i = 1,4.$$

By using the comitants R_i and S_i ($i = 1,4$), and the notion of transvectant the following $GL(2,\mathbb{R})$-comitants and invariants of the system (1) were constructed:

$$I_2 = (R_1,R_1)^{(2)},\quad K_1 = (S_4,R_1)^{(1)},\quad K_2 = ((S_4,R_1)^{(2)},R_1)^{(1)},$$
$$I_3 = (((S_4,R_1)^{(2)},R_1)^{(1)},(S_4,R_1)^{(2)})^{(1)},$$
$$I_4 = ((((R_4,R_1)^{(2)},R_1)^{(2)},R_1)^{(1)},((R_4,R_1)^{(2)},R_1)^{(2)})^{(1)},$$
$$K_3 = (R_4,S_4)^{(3)},\quad K_4 = (K_3^2,S_4)^{(3)},\quad K_5 = ((K_3,S_4)^{(2)},R_1)^{(2)},$$
$$I_5 = (((R_4,S_4)^{(2)},R_1)^{(2)},R_1)^{(2)},\quad I_6 = (K_4,K_5)^{(1)}$$

2 Lyapunov quantities for systems (1) with $S_1 = 0, I_2 \neq 0$

We will consider the system (1) with the conditions $S_1 = 0, I_2 > 0$, which has the center or focus at $(0,0)$. In these conditions system (1) can be reduced to the system

$$\frac{dx}{dt} = y + P_4(x,y), \qquad \frac{dy}{dt} = -x + Q_4(x,y), \tag{2}$$

and can be written [5] in the form

$$\frac{dx}{dt} = \frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{5}\frac{\partial R_4}{\partial y} + \frac{4}{5}S_4 x, \ \frac{dy}{dt} = -\frac{1}{2}\frac{\partial R_1}{\partial x} - \frac{1}{5}\frac{\partial R_4}{\partial x} + \frac{4}{5}S_4 y, \tag{3}$$

where $R_1 = x^2 + y^2$.

Let us consider the formal power series of the form

$$F(x,y) = x^2 + y^2 + \sum_{j=3}^{\infty} F_j(x,y)$$

where for each j, $F_j(x,y)$ is a homogeneous polynomial of degree j, so that the derivative of $F(x,y)$ along the solutions of the system (2) (or (3)) satisfies

$$\frac{dF(x,y)}{dt} = \sum_{k=2}^{\infty} G_{2k}(x^2+y^2)^k,$$

where G_{2k} are the polynomials of the coefficients of the system (2), called *Lyapunov quantities* [6].

For establishing the center conditions for the system (2) we will determine Lyapunov quantities. Polynomials $F_j(x,y)$ and constants G_{2k} can be determined from the infinite dimensional system of differential equations in partial derivatives:

$$(3m+2)(F_{3m+2}, R_1)^{(1)} + (3m-1)W(F_{3m-1}) = \\ = \begin{cases} 0, & \text{for } m = 2l-1,\ l \in \mathbb{N}^*, \\ G_{3m+2}R_1^{\frac{3m+2}{2}}, & \text{for } m = 2l,\ l \in \mathbb{N}^*, \end{cases} \tag{4}$$

where $F_2 = R_1$, $W(F_j) = (F_j, R_4)^{(1)} + \frac{4}{5}F_j S_4$.

From the system (4) it follows that only the homogeneous polynomials $F_{3m-1}(x,y)$, $m\in\mathbb{N}^*$ and the Lyapunov quantities G_{6l+2}, $l\in\mathbb{N}^*$ participate in solving the center-focus problem for the system (1). By solving consecutively the equations of system (4) the polynomials F_{3m-1} and respectively the Lyapunov quantities G_{6l+2}, are determined. The general form of the polynomials F_{3m-1}, $m\in\mathbb{N}^*$ and respectively, the general form of the Lyapunov quantities G_{6l+2}, $l\in\mathbb{N}^*$, are the following:

$$F_{3m+2} =$$

$$= \sum_{j=0}^{\left[\frac{3m+1}{2}\right]} \frac{(3m-1)\cdot(3m+2)!\cdot 2^{j+1}\cdot R_1^j\cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}, R_1)^{(1)}}{(3m-2j+1)!\cdot\prod\limits_{i=0}^{j}\left((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\right)},$$

$$G_{6l+2} = \frac{(6l-1)\cdot(6l+2)!\cdot 2^{3l+1}\cdot [\![W(F_{6l-1}), \overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{3l+1}}{\prod\limits_{i=0}^{3l}\left((6l-2i+2)^2\cdot(R_1,R_1)^{(2)}\right)}, \qquad (5)$$

where $m\in\mathbb{N}^*$, $l\in\mathbb{N}^*$, $W(F_i)=(F_i,R_4)^{(1)}+\frac{4}{5}F_iS_4$.

Noted that when $m=2l-1,\ l\in\mathbb{N}^*$, the respectively equations of the system (4) have a unique solution with respect to F_{3m+2}, i.e. in this case F_{3m+2} are determined unambiguously. In the case $m=2l,\ l\in\mathbb{N}^*$, the solutions of respectively equations of the system (4) with respect to F_{3m+2} are determined with accuracy to a term of the form $CR_1^{\frac{3m+2}{2}}$, where C is an arbitrary real constant. This implies that Lyapunov quantities $G_{6l+2},\ l\in\mathbb{N}^*$, are not determined unambiguously.

3 The $GL(2,\mathbb{R})$-invariant center conditions for a class of systems (1) with $S_1=0$, $I_2>0$, $I_3=I_4=0$

We will consider the system (3) (or (1)) with the conditions $S_1=0$,

$I_2 > 0$, which has the center or the focus at $(0,0)$.

If $R_4 \equiv 0$, then the system (3) (or (1)) with $S_1 = 0$ and $I_2 > 0$ has the singular point of the center type at the origin of coordinates. In this case the system (3) has the invariant algebraic curve

$$H(x,y) = 32R_1 \cdot K_2 + 8I_2 \cdot K_1 - 5I_2^2 = 0$$

and the first integral

$$|H|^{\frac{2}{3}} \cdot |R_1|^{-1} = c_1,$$

where c_1 is a real constant [7].

If $S_4 \equiv 0$, then the system (3) (or (1)) with $S_1 = 0$ and $I_2 > 0$ has the singular point of the center type at the origin of coordinates. In this case the system (3) has the first integral:

$$5R_1 + 2R_4 = c_2,$$

where c_2 is a real constant.

For the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ were established the $GL(2,\mathbb{R})$-invariant conditions for distinguishing between center and focus.

Theorem 1. *The system* (1) *with the conditions* $S_1 = 0$, $I_2 > 0$ *and* $I_3 = I_4 = 0$ *has the center in the origin of the coordinates if and only if the following conditions are fulfilled*

$$G_8 = G_{26} = G_{32} = G_{38} = 0,$$

where G_8, G_{26}, G_{32} *and* G_{38} *are Lyapunov quantities of respectively indices.*

Theorem 2. *The system* (1) *with the conditions* $S_1 = 0$, $I_2 > 0$ *and* $I_3 = I_4 = 0$ *has the center in the origin of the coordinates if and only if the following conditions are fulfilled*

$$I_5 = I_6 = 0.$$

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Integrability conditions for a cubic differential system with a bundle of two invariant straight lines and one invariant cubic

Dumitru Cozma, Anatoli Dascalescu

Abstract

For a cubic differential system with a bundle of two invariant straight lines and one invariant cubic it is proved that a weak focus is a center if and only if the first three Lyapunov quantities $L_j,\ j=\overline{1,3}$ vanish.

Keywords: Cubic differential system, center-focus problem, invariant algebraic curve, Lyapunov quantity, integrability.

1 Introduction

In this paper we consider the cubic system of differential equations

$$\begin{aligned}\dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y),\\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y),\end{aligned} \tag{1}$$

in which all variables and coefficients are assumed to be real. The origin $(0,0)$ is a singular point of a center or a focus type for (1), i.e. a weak focus. The goal of this paper is to solve the problem of the center for cubic system (1) with a bundle of two invariant straight lines and one invariant cubic curve.

The problem of the center for cubic system (1) with four invariant straight lines, three invariant straight lines, two invariant straight lines and one invariant conic was solved in [1]; with two parallel invariant straight lines and one invariant cubic was solved in [3].

2 The existence of a bundle of two invariant straight lines and one invariant cubic

Let the cubic system (1) have two invariant straight lines l_1, l_2 that are real or complex ($l_2 = \overline{l_1}$) intersecting at a point (x_0, y_0). The intersection point (x_0, y_0) is a singular point for (1) and has real coordinates. Without loss of generality we can take $l_1 \cap l_2 = (0,1)$. In this case the invariant straight lines can be written as

$$l_j \equiv 1 + a_j x - y = 0,\ a_j \in \mathbb{C},\ j = 1,2;\ a_2 - a_1 \neq 0. \tag{2}$$

In [2] it was proved that the straight lines (2) are invariant for (1) if and only if the following coefficient conditions are satisfied:

$$\begin{array}{l} f = -2,\ k = (a-1)(a_1+a_2) + g,\ l = -b,\ s = (1-a)a_1a_2, \\ m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1+a_2) - a + d + 2,\ r = 1, \\ n = -d - 1,\ p = b - c,\ q = (a_1 + a_2 - c)a_1a_2 - g. \end{array} \tag{3}$$

Next for cubic system (1) we find conditions for the existence of one invariant cubic passing through the same singular point $(0,1)$, i.e. forming a bundle with the lines l_1 and l_2 ($a_{03} = -1$).

$$\Phi(x,y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 - y^3 = 0, \tag{4}$$

where $(a_{30}, a_{21}, a_{12}) \neq 0$ and $a_{30}, a_{21}, a_{12} \in \mathbb{R}$.

The cubic (4) is an invariant curve for (1) if and only if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01} \in \mathbb{R}$ such that

$$P(x,y)\frac{\partial \Phi}{\partial x} + Q(x,y)\frac{\partial \Phi}{\partial y} \equiv \Phi(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y). \tag{5}$$

Identifying the coefficients of $x^i y^j$ in (5), we find that $c_{10} = 2a - a_{21}$, $c_{01} = a_{12} - 2b$, $d = (3a_{21} - 2a - 1)/2$, $g = (3a_{30} - 3a_{12} + 2b + 2c)/2$, $c_{11} = (5a_{21} + 2ca_{12} - 2a_{12}^2 + 3 - 6a)/2$, $c_{02} = 3b - a_{12}$, $c_{20} = [2c(a_{12}^2 + 2a_{21}) - 2a_{12}(a_{12}^2 + 3a_{21}) + 3a_{30} + +a_{12}(2a - 11 + 2(a_1+a_2)^2 - 2a_1a_2 - 2c(a_1+a_2)) + 6(b + c(a_1a_2 + 1) - a_1a_2(a_1+a_2))]/2$

and a_{30}, a_{21}, a_{12} are the solutions of the following system

$$
\begin{aligned}
F_{50} \equiv\ & -ca_{30}(a_{12}^2 + 2a_{21}) + (a_{21}a_1a_2 - a_{12}a_{30})(a-1)+ \\
& a_{30}(a_{12}^3 + 3a_{12}a_{21} + 3a_{30}) + 3a_{30}(a_1 + a_2)(a_1a_2 + a - 1)+ \\
& a_{12}a_{30}((a + 1 + a_2)(c - a_1 - a_2) + a_1a_2) - 3ca_{30}a_1a_2 = 0, \\
F_{41} \equiv\ & a_{12}^2(a_{12} + a_{30} - ca_{21}) + a_{21}^2(3a_{12} - 2c) + a_{30}(5a_{21} - ca_{12})+ \\
& 2a_{21}(a_1 + a_2)(a_1a_2 - 1 + a) - 2a_1a_2(ca_{21} - a_{12}(a - 1))- \\
& (a_{12}a_{21} + 3a_{30})(a - 1 - a_1a_2 + (a_1 + a_2)(a_1 + a_2 - c)) = 0, \\
F_{32} \equiv\ & a_{12}(a_1a_2 + a - 1)(a_1 + a_2) + a_1a_2(3 - 3a - ca_{12})- \\
& c(a_{12}^3 + 3a_{12}a_{21} + 3a_{30}) + a_{12}^4 + 4a_{12}^2a_{21} + 4a_{12}a_{30} + 2a_{21}^2- \\
& (a_{12}^2 + 2a_{21})(a - 1 - a_1a_2 + (a_1 + a_2)(a_1 + a_2 - c)) = 0, \\
F_{40} \equiv\ & 2a_{12}^2(a_{12} - c) + a_{21}(9a_{12} - a_{30} - 2b - 6c) - 2(b + c)+ \\
& a_{12}(5 - 2a + 2c(a_1 + a_2) - 2(a_1 + a_2)^2 + 2a_1a_2)+ \\
& 4(a_1 + a_2)(a - 1) + 6a_1a_2(a_1 + a_2 - c) + a_{30}(2a + 3) = 0, \\
F_{31} \equiv\ & 2a(a_{21} + 2a_1a_2 - 1) + (a_{12} - a_{30})(8a_{12} - 4b) - a_{21}^2+ \\
& 2c(2a_1 + 2a_2 - 3a_{12} + 3a_{30}) - 4(a_1 + a_2)^2 + 2a_{21} + 3 = 0, \\
F_{22} \equiv\ & a_{12}^2(c - a_{12}) + a_{12}((a_1 + a_2)^2 - a_1a_2 - c(a_1 + a_2))- \\
& b(a_{21} + 1) - a_1a_2(a_1 + a_2 - c) = 0.
\end{aligned}
\tag{6}
$$

Let us denote $j_1 = a_{12}(a_1 + a_2) - 3a_1a_2 - a_{12}^2 - 2a_{21}$, $j_2 = a_2^3 - a_2^2a_{12} - a_2a_{21} - a_{30}$, $j_3 = a_1^3 - a_1^2a_{12} - a_1a_{21} - a_{30}$, $j_4 = 4a_{12}^3a_{30} - a_{12}^2a_{21}^2 + 18a_{12}a_{21}a_{30} - 4a_{21}^3 + 27a_{30}^2$.

Theorem 1. *The system of algebraic equations (6) is compatible if and only if* $j_1j_2j_3j_4 = 0$.

We study the compatibility of (6) when $a_1 - a_2 \neq 0$, $a - 1 \neq 0$, $(a_{30}, a_{21}, a_{12}) \neq 0$ and divide the investigation into four cases:
$\{j_1 = 0\}$, $\{j_1 \neq 0, j_2 = 0\}$, $\{j_1j_2 \neq 0, j_3 = 0\}$, $\{j_1j_2j_3 \neq 0, j_4 = 0\}$.

The solutions of (6) with respect to a_{30}, a_{21}, a_{12} give us the conditions under which the cubic system (1) has at least one irreducible invariant cubic. There were obtained 22 sets of necessary and sufficient conditions for cubic system (1) to have a bundle of two invariant straight lines and one invariant cubic.

We compute the first three Lyapunov quantities for each set of conditions and establish the cyclicity of the weak focus.

3 The problem of the center

The solution of the problem of the center for cubic differential system (1) with a bundle of two invariant straight lines and one invariant cubic, is given in the following two theorems:

Theorem 2. *Let the cubic system (1) have a bundle of two invariant straight lines $l_1 = 0$, $l_2 = 0$ and one invariant cubic $\Phi = 0$, then* $(0,0)$ *is a center if and only if the first three Lyapunov quantities vanish.*

Theorem 3. *The cubic system (1) with a center having a bundle of two invariant straight lines and one invariant cubic is Darboux integrable.*

Assuming that the first three Lyapunov quantities vanish, there were obtained 18 sets of conditions for $(0,0)$ to be a center.

4 Conclusion

For a cubic differential system with a weak focus, having a bundle of two invariant straight lines and one invariant cubic, there were obtained the conditions under which the weak focus is a center.

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Duality and a Riemann metrics in theory of a second order ODE's

Valerii Dryuma

Abstract

A properties of integral curves of ODE's of the second order $\frac{d^2}{dx^2}y(x) = Q(x, y, y')$ and of the first order $\frac{d}{dx}y(x) = \frac{Q_n(x,y)}{P_n(x,y)}$ with a help of conformal $4D$- Riemann metrics of Fefferman and Walker types are investigated.

Keywords: CR-structures, Duality, Fefferman metrics, Bach tensor, Walker metrics).

1 Introduction

Geometric methods play an important role in theory of differential equations. Here we deal with geometry of a second order ODE $y'' = Q(x, y, y')$ considered modulo point transformations of variables. Using analogy between the $2d$- order ODE's and 3-dimensional CR-structures and theory of duality developed by E.Cartan we apply conformal $4D$-metrics of Fefferman with signature $(2, 2)$ to the study of Invariant properties of integral curves of a second order ODE's having the form $y'' = A_4(x, y) + A_3(x, y)y') + A_2(x, y)y'^2 + A_1(x, y)y'^3$ and of a more complicate type. In particular properties of algebraic second order ODE's $F(y, y', y'') = 0$ and $F(x, y', y'') = 0$ which admit reduction to a first order ODE's $H(x, y, y') = 0$ are considered. With this aim Bach-tensor B_{ik} of the metric expressed through the derivatives of the Weyl tensor C_{ijkl} is applied.

2 Duality to a second order ODE's

The relation between a four variables $F(x, y, a, b) = 0$ determines some $3D$-manifold and generates two a second order ODE's $y'' = f(x, y, y')$ and $b'' = h(a, b, b')$. A first one is result of elimination of variables a, b from the conditions

$$F(x, y, a, b) = 0,\ F_x + y'F_y = 0,\ F_{xx} + 2y'F_{xy} + F_{yy}y'^2 + F_y y'' = 0,$$

and a second one is obtained after eliminating of the variables x, y from the system

$$F(x, y, a, b) = 0,\ F_a + b'F_b = 0,\ F_{aa} + 2b'F_{ab} + F_{bb}b'^2 + F_b b'' = 0.$$

Taking into account that only general integral $F(x, y, a, b) = 0$ of an arbitrary second-order equation $y'' = f(x, y, y')$ contains complete information about solutions E. Cartan developed geometric theory of Duality for various invariant classes of the second order ODE's. As example the equations of the form $y'' = A_4(x, y) + A_3(x, y)y' + A_2(x, y)y'^2 + A_1(x, y)y'^3$ and $b'' = h(a, b, b')$ where the function $h(a, b, b' = c)$ is solution of the p.d.e. $D^2h_{cc} - 4Dh_{bc} - h_cDh_{cc} + 4h_ch_{bc} - 3h_bh_{cc} + 6h_{bb} = 0$, where $D = \partial_a + c\partial_b + h\partial_c$ form a dual pairs.

Theorem 1. *To the equation* $(b_0 + b_1y' + b_2y'^2)y'' = a_0 + a_1y' + a_2y'^2 + a_3y'^3 + a_4y'^4 + a_5y'^5$, $b = b_i(x, y), a_i = a_i(x, y)$, *from the point invariant class* $h_m(x, y, y')y'' = h_{m+3}(x, y, y'), m = 2$ *dual equation determined by solutions of the equation* $\psi_6^3 + \psi_8\psi_5^2 + \psi_4\psi_7^2 - \psi_4\psi_5\psi_8 - 2\psi_5\psi_6\psi_7 = 0$, *where* $4!\psi_4 = -\frac{d^2}{da^2}h_{cc} + 4\frac{d}{da}h_{bc} + h_c\frac{d}{da}h_{cc} - 4h_ch_{bc} + 3h_bh_{cc} - 6h_{bb}$ *and* $\psi_6^2 - \psi_4\psi_5 = 0, m = 1$, *where* $k\psi_k = \frac{d}{da}\psi_{k-1} + (3 - k)h_c\psi_{k-1} + (k - 1) + h_b\psi_{k-2}, k > 4$.

Example 1 *The equations*

$$-4 + 3x^4\, y(x)^2 + 2\, x^5 y'y'' = 0, b'' + \frac{1}{a}\tan(1/2\, b(a)) = 0$$

with General Integral

$$F(x, y, a, b) = b + 2\ \arctan(\frac{1}{\sqrt{-1 + xa}})a + 2\,\frac{\sqrt{-1 + xa}}{x} - y = 0$$

form a dual pair.

3 Fefferman metrics

Definition 1 [2]. *The Fefferman metrics is the metrics on* 4*-manifolds in coordinates* $(x, y, p = y')$ *associated with an second order ODE's* $y'' = Q(x, y, y')$*, considered modulo point transformation* $x = f(x, y), y = g(x, y)$*.* It has the form

$$ds^2 = (dz - Qdx)dx - (dy - zdx)(dw + \frac{2}{3}Q_z dx + \frac{1}{6}Q_{zz}(dy - zdx))$$

Definition 2 . *Bach tensor of* 4*-manifolds has the form*

$$B_{ik} = \nabla^r \nabla^s C_{risk} + \frac{1}{2} R^{rs} C_{irks},$$

where C_{ijkl} is Weyl tensor, R^{ij} is Ricci tensor of the metrics.

Theorem 2. *For the conformal Fefferman metrics*

$$-e^{-A(x,y,z)} ds^2 = -6\, l dz dx + 6\, l Q(x, y, z) dx^2 + dy dw +$$
$$+4l \left(\frac{\partial}{\partial z} Q(x, y, z)\right) dx\, dy + l \left(\frac{\partial^2}{\partial z^2} Q(x, y, z)\right) dy^2 -$$
$$-2l \left(\frac{\partial^2}{\partial z^2} Q(x, y, z)\right) z dx\, dy - z dx dw - 4l \frac{\partial}{\partial z} Q(x, y, z)\, z dx^2 +$$
$$+z^2 l \frac{\partial^2}{\partial z^2} Q(x, y, z) dx^2$$

all components of Bach tensor equal to zero $B_{xx} = 0, B_{xy} = 0, B_{yy} = 0$ if the function $Q(x, y, z)$ satisfies to the p.d.e.

$$2\, Q Q_{zzzz} Q_{zz} + z^2 Q_{yyzzzz} + 2z\, Q_{xyzzzz} + Q_{xxzzzz} + Q^2 Q_{zzzzzz} + z Q_y Q_{zzzzz} +$$
$$+2z\, Q(x, y, z) Q_{yzzzzz} + 2\, Q_z^2 Q_{zzzz} + 3\, Q_z Q_{xzzzz} + 2\, Q Q_{xzzzz} +$$
$$+2z\, Q_{yz} Q_{zzzz} + 2\, Q_{xz} Q_{zzzz} + Q_x Q_{zzzzz} + 3z\, Q_z Q_{yzzzz} +$$
$$+4\, Q Q_z Q_{zzzzz} + Q Q_{yzzzz} - Q_y Q_{zzzz} = 0. \qquad (1)$$

Corollary 1 *Class of the second order ODE's* $y'' = Q(x, y, y')$ *with the function* $Q(x, y, y')$ *from [1] contains the equations of the form* $y'' = A_4(x, y) + A_3(x, y) y' + A_2(x, y) y'^2 + A_1(x, y) y'^3$ *and dual of them.*

4 Conformal Walker metrics

Definition 3 [3] . *Conformal Walker metrics on 4-manifolds in coordinates* (x, y, z, w) *associated with the second order ODE's* $y'' = a_4(x,y)+3a_3(x,y)y'+3a_2(x,y)y'^2+a_1(x,y)y'^3$, *considered modulo point transformation* $x = f(x,y), y = g(x,y)$ *has the form*

$$\frac{1}{2}\,e^{-A(x,y)}\,ds^2 = dx\,dz + dy\,dw + 2\,(\,za_2(x,y) - \,wa_3(x,y))\,dx\,dy+$$
$$+\,(za_3(x,y) - a_4(x,y)w)\,dx^2 + (za_1(x,y) - wa_2(x,y))\,dy^2. \qquad (2)$$

This type of metric generalizes the standard Walker metric and is applied to the study of the properties of geodesics and transfer surfaces defined by the system of equations $Z^i_{u,v} + \Gamma^i_{jk} Z^j_u Z^k_v = 0$, where Γ^i_{jk} are Christoffel symbols of the metrics.

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Stable Spectral Collocation Solutions to Cauchy Problems for Nonlinear Dispersive Wave Equations

Călin-Ioan Gheorghiu

Abstract

In this paper we are concerned with accurate and stable spectral collocation solutions to initial-boundary value problems attached to some challenging nonlinear wave equations defined on unbounded domains. We argue that spectral collocation based on Hermite and sinc functions actually provide such solutions avoiding the empirical domain truncation or any shooting techniques.

Keywords: Hermite, sinc, collocation, nonlinear, wave equation, shock like solution.

1 Introduction

The most useful technique to solve initial-boundary value problems attached to nonlinear parabolic equations on unbounded domains (half line or the real line) involve the truncation of the domain to a finite computational one, say $[x_L, x_R]$, with approximate boundary conditions imposed at $x = x_L$ and $x = x_R$. One of the most difficult numerical issue for such technique is the *sensitivity of a correct numerical solution* to the appropriate boundary conditions, especially the one imposed at the right-hand boundary. In order to avoid this tedious discussion on the proper boundary conditions at the ends of the computational domain we will try to solve the aforementioned problems by Hermite collocation (HC) and sinc collocation (SiC).

In [1] the author observes that for problems on unbounded domains boundary conditions are usually "natural" rather than "essential" in the sense that the singularities of the differential equation will force the numerical solution to have the correct behavior at infinity even if no constraints are imposed on the basis functions. In this respect our initial-boundary value problems reduce to some Cauchy problems.

2 Nonlinear wave equations with linear dispersion

We are concerned with non-periodic *spectral collocation solutions* for *initial value problems* attached to *nonlinear wave equations* of the form

$$u_t = N(u) + L(u) + g(x,t), \ -\infty \leq x \leq \infty, \ t \geq 0. \tag{1}$$

The term $N(u)$ is a genuinely nonlinear one and may also depend on u_x, u_{xx}, etc. and the linear part is of the form

$$L(u) := c(t)\, i^{m+1} \left(\partial^m u/\partial x^m\right), \tag{2}$$

but more general *dispersive terms* are also treatable. The *forcing term* $g(x,t)$ can be embedded into $N(u)$. The real function $c(t)$ is often a constant.

The Benjamin Bona Mahony (BBM) type problems have been considered in our previous paper [4]. The Korteweg-de Vries (KdV), the nonlinear Schrödinger (NLS) and the Fisher's initial value problems are more challenging examples of such problems which will be partially addressed now.

3 Numerical analysis

The spectral collocation is based alternatively on the *scaled Hermite* and *sinc functions*. This spatial discretization approach avoids periodicity (see [2]) and frequently used empirical domain truncation (see

[3]). In order to march in time we use TR-BDF2 (`ode23tb` built in routine in MATLAB), the trapezoidal rule using a "free" interpolant (`ode23t`) and a modified Rosenbrock formula of order 2 (`ode23s`) FD schemes. We show that the method of lines (MoL) involved is *stable* using the *pseudospectra* of the linearized spatial discretization operators (see also [6]). The sinc collocation along with TR-BDF2 perform better than the other methods with respect to the accuracy and the computational effort. A heuristic explanation is provided.

The extent at which some *invariants* are conserved over time has been analyzed in our contribution [5]. It also proved to be fairly useful in optimizing the scaling parameters.

3.1 Fisher's equation with nonlocal boundary conditions

To be more specific, we define $L(u) := u_{xx}$ and $N(u) := \rho u(1-u)$ in (1) where ρ stands for the *reaction factor*, i.e. the Fisher's equation. We are mainly interested in super speed waves (SSW). With an increase in ρ, the propagating front steepens and this presents a challenging numerical problem in order to resolve as well as to track the front. For the infinite spatial domain, the rapidly varying shock front is considered to be stiff with the stiffness depending on ρ.

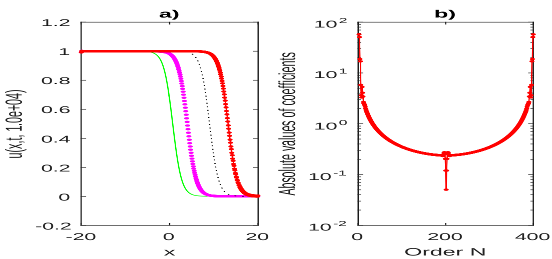

Figure 1. a) SSW time dependent profiles of Fisher's equation at $t = 0$, $t = 3.3065e - 04$, $t = 8.3189e - 04$ and $t = 0.0013$. b) The absolute values of the expansion coefficients of the solution at $t = 0.0013$.

4 Conclusion

The effectiveness of our approach has been confirmed by some challenging numerical experiments. Using SiC along with TR-BDF2 we have succeeded in capturing shock like solutions to Fisher's problem (see Fig. 3.1 a)). With FFT we get the expansion coefficients of the final solution. Their decreasing behavior is reported in Fig. 3.1 b).

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Topological mixing and specification in weakly contracting relations

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Abstract

We establish topological mixing and Specification property of the dynamics of a weakly contracting compact-valued function on its attractor. These properties strengthen our earlier results, obtained for weakly contracting relations, namely, the topological transitivity and Shadowing property.

Keywords: Set-valued maps, weak contractions, attractor, shadowing, specification property, topological mixing.

1 Introduction

In [1] the authors have stated the existence of the compact global attractor for a relation, which is contracting with respect to the Hausdorff-Pompeiu metrics. Moreover, some characteristics of set-valued dynamics of these relations restricted to their attractors, have been stated, as, e.g., "asymptotic phase property", topological transitivity, denseness of periodic points, minimality with respect to "big orbits", and Shadowing property. In [2] some of these properties, including "asymptotic phase property", topological transitivity and Shadowing have been generalized for weakly contracting multi-functions. In this article we strengthen the last two properties up to topological mixing and Specification property, respectively.

2 Multi-valued weak contractions

Let (X, d) be a complete metric space and let $\mathcal{P}(X)$ denote the family of all non-empty compact subsets, endowed with Hausdorff-Pompeiu metrics H. We are concerned with the dynamics, generated by compositions of upper-semi-continuous multi-functions $f : X \to \mathcal{P}(X)$, called also as *relations*. In this context, a finite or infinite sequence $\{x_n\} \subset X$ is called a *chain* for the multi-function f, if $x_{n+1} \in f(x_n)$ for all n. Similarly, given $\delta > 0$, the sequence $\{x_n\}$ is called a δ-*chain*, if $\varrho(x_{n+1}, f(x_n)) \leq \delta$ for all n (here $\varrho(a, B) := \inf\limits_{b \in B} d(a, b)$).

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* [3], if φ is monotonically increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \geq 0$.

Following [3], we will say that $f : X \to \mathcal{P}(X)$ is a *weak contraction*, if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $H(f(x), f(y)) \leq \varphi(d(x, y))$ $(\forall\, x, y \in X)$.

A nonempty closed subset $A \subset X$ is called *attractor* for f, if $f[A] \supset A$ and there is a closed neighborhood $\overline{V(A, \delta)}$ of A, where $V(A, \delta) := \{x \in X \mid \varrho(x, A) < \delta\}$, such that $\bigcap\limits_{n \geq 0} f^n[\overline{V(A, \delta)}] \subset A$.

One says that the relation $f : X \to \mathcal{P}(X)$ has the *Shadowing property* on the compact invariant subset $A \subset X$ if, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-chain $\{x_n\}_{n \in \mathbb{N}} \subset V(A, \delta)$ there exists a chain $\{y_n\}_{n \in \mathbb{N}} \subset A$ satisfying $d(x_n, y_n) \leq \varepsilon$ for all $n \in \mathbb{N}$.

Theorem 1. [2] *Any weakly contracting compact valued mapping has a nonempty compact attractor and this attractor is unique. If, in addition, f is weakly contracting with respect to a right-continuous comparison function, then the multi-function f has the Shadowing property on the attractor.*

3 Topological mixing and specification

Recall (see, e.g. [1]) that a multi-function $f : X \to \mathcal{P}(X)$ is called *transitive* on the compact invariant subset A, if there is a dense chain, or equivalently, if for any two open subsets $U, V \subset A$ and any $x \in U$ there is a chain $(x_k)_{k=1}^n \subset A$ such that $x_1 = x$ and $x_k \in V$. If, in addition, there is a chain $(x_n)_{n\in\mathbb{N}} \subset A$, which starts in U and which remains in V for all large enough n, then one speaks about *topological mixing*. Topological transitivity of a contracting relation on its attractor has been stated in [2]. In this article we establish stronger analogous of transitivity and Shadowing properties, namely topological mixing and specification.

Theorem 2. *Every multi-function, which is weakly contracting with respect to a right-continuous comparison function, is topologically mixing on its attractor.*

The Specification property for diffeomorphisms was introduced by R. Bowen [4]. It says that any finite collection of consecutive pieces of orbits of $f : X \to X$ can be shadowed by an individual orbit, provided that the time-lag between the specified orbit segments is large enough.

The following definition represents a generalization of the Specification property, given in [5] for homeomorphisms (see also [6, 7]). More precisely, given the multi-function $f : X \to \mathcal{P}(X)$, we call *specification* for f the pair $S = (\tau, P)$, consisting of a finite family of time-segments $\tau = \{I_1, I_2, \ldots, I_m\}, I_j \subset \mathbb{N}$, and a mapping $P : \bigcup_{j=1}^{m} I_j \to X$, such that $P(t+1) \in f(P(t))$ for all $t \in I \in \tau$, provided that $t+1 \in I$. We say that the specification S is ε-*shadowed* by the chain $(x_n)_{n\in\mathbb{N}}$, if $d(x_n, P(n)) < \varepsilon$ for all $n \in \bigcup_{j=1}^{m} I_j$. Given the natural number M, the specification is called $M-$*spaced*, if the gap between two consecutive time-segments is at least M.

One says that the multi-function f has *the Specification property* on the invariant compact subset $A \subset X$ if, given $\varepsilon > 0$, there is a natural M such that each $M-$spaced specification is ε-shadowed by a chain.

Theorem 3. *Every multi-function, which is weakly contracting with respect to a right-continuous comparison function, has the Specification property on its attractor.*

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Positive solutions for a system of difference equations with coupled boundary conditions

Johnny Henderson, Rodica Luca

Abstract

We study the existence and multiplicity of positive solutions for a system of nonlinear second-order difference equations subject to coupled multi-point boundary conditions.

Keywords: difference equations, coupled multi-point boundary conditions, positive solutions.

1 Introduction

We consider the system of nonlinear second-order difference equations

$$(S) \qquad \begin{cases} \Delta^2 u_{n-1} + f(n, v_n) = 0, & n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + g(n, u_n) = 0, & n = \overline{1, N-1}, \end{cases}$$

with the coupled multi-point boundary conditions

$$(BC) \qquad u_0 = 0, \ \ u_N = \sum_{i=1}^{p} a_i v_{\xi_i}, \ \ v_0 = 0, \ \ v_N = \sum_{i=1}^{q} b_i u_{\eta_i},$$

where $N \in \mathbb{N}$, $N \geq 2$, p, $q \in \mathbb{N}$, Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}$, $n = \overline{k, m}$ means that $n = k, k+1, \ldots, m$ for k, $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$ and $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$.

Under sufficient conditions on the functions f and g, we study the existence and multiplicity of positive solutions of problem $(S) - (BC)$

by using some theorems from the fixed point index theory. By a positive solution of problem $(S)-(BC)$ we mean a pair of sequences $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ satisfying (S) and (BC), with $u_n \geq 0$, $v_n \geq 0$ for all $n = \overline{0,N}$ and $(u,v) \neq (0,0)$.

The existence of positive solutions for system (S) with two parameters λ and μ (denoted by (S_1)), with the coupled boundary conditions (BC) was investigated in [1]. The systems (S_1) and (S) subject to some uncoupled boundary conditions have been studied in [2] by using the Guo-Krasnosel'skii fixed point theorem, the fixed point index theory and the Schauder fixed point theorem.

2 Main results

We present the basic assumptions that we shall use in the sequel.

$(A1)$ $a_i \geq 0$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1,p}$, $b_i \geq 0$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1,q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$ and $\Delta_0 = N^2 - \left(\sum_{i=1}^p a_i\xi_i\right)\left(\sum_{i=1}^q b_i\eta_i\right) > 0$.

$(A2)$ The functions f, $g : \{1,\dots,N-1\} \times [0,\infty) \to [0,\infty)$ are continuous.

By using the associated Green functions G_i, $i = \overline{1,4}$ (see [1]), our problem $(S)-(BC)$ can be written equivalently as the following system

$$\begin{cases} u_n = \displaystyle\sum_{i=1}^{N-1} G_1(n,i) f(i,v_i) + \sum_{i=1}^{N-1} G_2(n,i) g(i,u_i), \quad n = \overline{0,N}, \\ v_n = \displaystyle\sum_{i=1}^{N-1} G_3(n,i) g(i,u_i) + \sum_{i=1}^{N-1} G_4(n,i) f(i,v_i), \quad n = \overline{0,N}. \end{cases}$$

We consider the Banach space $X = \mathbb{R}^{N+1} = \{u = (u_0, u_1, \dots, u_N),\ u_i \in \mathbb{R},\ i = \overline{0,N}\}$ with the maximum norm $\|\cdot\|$, $\|u\| = \max_{n=\overline{0,N}} |u_n|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cone $P \subset Y$ by $P = \Big\{(u,v) \in Y;\ u = (u_n)_{n=\overline{0,N}},$
$v = (v_n)_{n=\overline{0,N}},\ u_n \geq 0,\ v_n \geq 0,\ \forall\, n = \overline{0,N}\Big\}$.

We introduce the operators Q_1, $Q_2 : Y \to X$ and $\mathcal{Q} : Y \to Y$ defined by $Q_1(u,v) = (Q_1(u,v))_{n=\overline{0,N}}$, $Q_2(u,v) = (Q_2(u,v))_{n=\overline{0,N}}$,

$$(Q_1(u,v))_n = \sum_{i=1}^{N-1} G_1(n,i) f(i,v_i) + \sum_{i=1}^{N-1} G_2(n,i) g(i,u_i),\ n = \overline{0,N},$$

$$(Q_2(u,v))_n = \sum_{i=1}^{N-1} G_3(n,i) g(i,u_i) + \sum_{i=1}^{N-1} G_4(n,i) f(i,v_i),\ n = \overline{0,N},$$

and $\mathcal{Q}(u,v) = (Q_1(u,v), Q_2(u,v))$, $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}) \in Y$.

Under the assumptions $(A1)$ and $(A2)$, it is easy to see that the operator $\mathcal{Q} : P \to P$ is completely continuous (see also Lemma 3.1 from [1]). Thus the existence and multiplicity of positive solutions of problem $(S)-(BC)$ are equivalent to the existence and multiplicity of fixed points of operator $\mathcal{Q}$.

Our main existence results for problem $(S)-(BC)$ are the following theorems.

Theorem 1 [3]. *Assume that $(A1)$ and $(A2)$ hold. If the functions f and g also satisfy the conditions*

$(A3)$ *There exists $c \in \{1, \ldots, [\![N/2]\!]\}$ such that*

$$f_\infty^i = \lim_{u\to\infty} \min_{n=\overline{c,N-c}} \frac{f(n,u)}{u} = \infty,\ g_\infty^i = \lim_{u\to\infty} \min_{n=\overline{c,N-c}} \frac{g(n,u)}{u} = \infty,$$

$(A4)$ *There exist $p_1 \geq 1$ and $q_1 \geq 1$ such that*

$$f_0^s = \lim_{u\to0^+} \max_{n=\overline{1,N-1}} \frac{f(n,u)}{u^{p_1}} = 0,\ g_0^s = \lim_{u\to0^+} \max_{n=\overline{1,N-1}} \frac{g(n,u)}{u^{q_1}} = 0,$$

then problem $(S)-(BC)$ has at least one positive solution $((u_n)_{n=\overline{0,N}}$, $(v_n)_{n=\overline{0,N}})$.

Theorem 2 [3]. *Assume that $(A1)$ and $(A2)$ hold. If the functions f and g also satisfy the conditions*

$(A5)$ $f_\infty^s = \lim_{u\to\infty} \max_{n=\overline{1,N-1}} \frac{f(n,u)}{u} = 0,\ g_\infty^s = \lim_{u\to\infty} \max_{n=\overline{1,N-1}} \frac{g(n,u)}{u} = 0,$

$(A6)$ *There exist $c \in \{1, \ldots, [\![N/2]\!]\}$, $p_2 \in (0,1]$ and $q_2 \in (0,1]$ such that*

$$f_0^i = \lim_{u\to0^+} \min_{n=\overline{c,N-c}} \frac{f(n,u)}{u^{p_2}} = \infty,\ g_0^i = \lim_{u\to0^+} \min_{n=\overline{c,N-c}} \frac{g(n,u)}{u^{q_2}} = \infty,$$

then problem $(S)-(BC)$ *has at least one positive solution* $((u_n)_{n=\overline{0,N}}$, $(v_n)_{n=\overline{0,N}})$.

Theorem 3 [3]. *Assume that* $(A1)-(A3)$ *and* $(A6)$ *hold. If the functions* f *and* g *also satisfy the condition*

$(A7)$ *For each* $n=\overline{1,N-1}$, $f(n,u)$ *and* $g(n,u)$ *are nondecreasing with respect to* u, *and there exists a constant* $R_0>0$ *such that*

$$f(n,R_0)<\tfrac{R_0}{4m_0},\ \ g(n,R_0)<\tfrac{R_0}{4m_0},\ \ \forall\, n=\overline{1,N-1},$$

where $m_0=\max\{M_i,\ i=\overline{1,4}\}$, *(*$M_i=\sum_{j=1}^{N-1}I_i(j)$, $i=\overline{1,4}$, *and* I_i, $i=\overline{1,4}$ *are defined in [1]), then problem* $(S)-(BC)$ *has at least two positive solutions* $((u_n^1)_{n=\overline{0,N}},(v_n^1)_{n=\overline{0,N}})$ *and* $((u_n^2)_{n=\overline{0,N}},(v_n^2)_{n=\overline{0,N}})$.

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Simultaneous Approximation in Weighted Lebesgue Spaces with Variable Exponent

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Abstract

The higher fractional order modulus of smoothness is defined and this term the simultaneous approximation of trigonometric and near-best approximating polynomials in the weighted variable exponent Lebesgue spaces are investigated.

Keywords: Simultaneous theorems, Muckenhoupt weights, fractional order modulus of smoothness, variable spaces.

1 Introduction

Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [0, \infty)$ be a Lebesgue measurable 2π periodic function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is defined as the set of all Lebesgue measurable 2π periodic functions f such that $\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx < \infty$. During this work we suppose that the considered exponent functions $p(\cdot)$ satisfy the conditions

$$1 \leq p_- := ess\inf_{x\in\mathbb{T}}\ p(x) \leq ess\sup_{x\in\mathbb{T}}\ p(x) := p^+ < \infty,$$
$$|p(x) - p(y)| \ln(1/|x-y|) \leq c(p) < \infty, \quad x, y \in \mathbb{T}, \quad 0 < |x-y| \leq 1/2.$$

The class of these exponents we denote by $\mathcal{P}(\mathbb{T})$. If $p(\cdot) \in \mathcal{P}(\mathbb{T})$ and in addition $p_- > 1$, then we say that $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. Equipped with the norm $\|f\|_{p(\cdot)} = \left\{\inf \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\right\}$ the space $L^{p(\cdot)}(\mathbb{T})$ becomes a Banach space. Let ω be a weight function on $\mathbb{T}$, i.e. an almost everywhere positive and Lebesgue integrable function on $\mathbb{T}$. For a given weight ω we define the weighted variable exponent Lebesgue space $L^{p(\cdot)}_{\omega}(\mathbb{T})$as the set of all measurable functions f on $\mathbb{T}$ such that $f\omega \in L^{p(\cdot)}(\mathbb{T})$. The norm of $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$ can be defined as $\|f\|_{p(\cdot),\omega} := \|f\omega\|_{p(\cdot)}$. In our discussions we assume that $\omega \in A_{p(\cdot)}(\mathbb{T})$.

Definition 1 *We say that* $\omega \in A_{p(\cdot)}(\mathbb{T})$ *if the inequality*

$$\sup_{I \subset \mathbb{T}} |I|^{-1} \|\omega \chi_I\|_{p(\cdot)} \left\|\omega^{-1}\chi_I\right\|_{p'(\cdot)} < \infty, \; 1/p(\cdot) + 1/p'(\cdot) = 1,$$

holds, where $|I|$ *is the Lebesgue measure of the interval* $I \subset \mathbb{T}$ *with the characteristic function* χ_I.

Let $f \in L^1(\mathbb{T})$ with $\int_0^{2\pi} f(x)\,dx = 0$. For $\alpha \in \mathbb{R}^+$ the αth integral of f is defined by $I_\alpha(f,x) := \sum\limits_{k\in\mathbb{Z}^*} c_k(f)(ik)^{-\alpha} e^{ikx}$, where $(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i\alpha \, sign\, k}$, $\mathbb{Z}^* := \{\pm1, \pm2, \pm3, ...\}$ and c_k, $k \in \mathbb{Z}^*$, are the Fourier coefficients of f with respect to exponential system. For $\alpha \in (0,1)$ let $f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(f,x)$. If $r \in \mathbb{R}^+$ with integer part $[r]$, and $\alpha := r - [r]$, then the rth derivative of f is defined by $f^{(r)}(x) := \left(f^{(\alpha)}(x)\right)^{([r])} = \frac{d^{[r]+1}}{dx^{[r]+1}} I_{1-\alpha}(f,x)$ if the right sides exist [1, p. 347]. Let $x, t \in \mathbb{R}$, $r \in \mathbb{R}^+$ and let $\Delta_t^r f(x) := \sum_{k=0}^{\infty} (-1)^k [C_k^r] f(x + (r-k)t)$ for $f \in L^1(\mathbb{T})$, where $[C_k^r] := r(r-1)(r-2)...(r-k+1)/k!$ for $k > 1$, $[C_k^r] := r$ for $k = 1$ and $[C_k^r] := 1$ for $k = 0$.

Definition 2 *Let* $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ *and* $r \in \mathbb{R}^+$. *We define the* r*th modulus of smoothness as*

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \sup_{|h|\le\delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r f(x)\,dt \right\|_{p(\cdot),\omega}, \quad \delta > 0.$$

Clearly $\Omega(f,\delta)_{p(\cdot),\omega}$ is well defined because by Theorem on the boundedness of maximal function in $L^{p(\cdot)}_{\omega}(\mathbb{T})$ proved in [2] we have $\Omega_r(f,\delta)_{p(\cdot),\omega} \le c\|f\|_{p(\cdot),\omega}$.

By $S_n(f)$ we denote the nth partial sum of the Fourier series of $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$. Let $W^{p(\cdot)}_{\omega,\beta}(\mathbb{T}) := \left\{ f \in L^{p(\cdot)}_{\omega}(\mathbb{T}) : \; f^{(\beta)} \in L^{p(\cdot)}_{\omega}(\mathbb{T}) \text{ for } \beta > 0 \right\}$ be the weighted variable exponent Sobolev space.

2 Main Results

By $c(p)$, $c_1(p)$, ... we denote the different constants depending in general of the parameters given in the brackets but independent of n.

Let $E_n(f)_{p(\cdot),\omega} := \inf\left\{\|f - T_n\|_{p(\cdot),\omega} : T_n \in \Pi_n\right\}$. $n \in \mathbb{N}$, where Π_n is the class of trigonometric polynomials of degree not exceeding n and let $T_n^* := T_n^*(f)$ be the *near-best approximating* polynomial to f in Π_n, i.e., $\|f - T_n^*\|_{p(\cdot),\omega} \leq c(p)E_n(f)_{p(\cdot),\omega}$, for some constant $c(p) > 0$, independent of n.Our main results are following :

Theorem 1 *Let* $f \in W^{p(\cdot)}_{\omega,k}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega \in A_{p(\cdot)}(\mathbb{T})$ *and* $k \in \mathbb{R}^+$. *If* $T_n^* \in \Pi_n$ *is a near-best approximating polynomial to* f, *then the inequality* $\left\|f^{(k)} - (T_n^*)^{(k)}\right\|_{p(\cdot),\omega} \leq c(p,k)E_n\left(f^{(k)}\right)_{p(\cdot),\omega}$ *holds.*

Theorem 2 *Let* $f \in W^{p(\cdot)}_{\omega,m}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega \in A_{p(\cdot)}(\mathbb{T})$ *and* $r, m \in \mathbb{R}^+$. *If*

$$\|f - T_n\|_{p(\cdot),\omega} \leq \frac{c(p)}{n^m}\Omega_r\left(f^{(m)}, 1/n\right)_{p(\cdot),\omega}, \qquad n = 1, 2, ...$$

for a trigonometric polynomial $T_n \in \Pi_n$, *then for every* $k \in \mathbb{R}^+$ *with* $0 < k \leq m$

$$\left\|f^{(k)} - T_n^{(k)}\right\|_{p(\cdot),\omega} \leq \frac{c(p,k)}{n^{m-k}}\Omega_r\left(f^{(m)}, 1/n\right)_{p(\cdot),\omega}.$$

Theorem 3 *If* $f \in W^{p(\cdot)}_{\omega,m}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega \in A_{p(\cdot)}(\mathbb{T})$ *and* $r, m \in \mathbb{R}^+$, *then for every* $k \in \mathbb{R}^+$ *with* $0 < k \leq m$

$$\left\|f^{(k)} - S_n^{(k)}(f)\right\|_{p(\cdot),\omega} \leq \frac{c(p,k,r)}{n^{m-k}}\Omega_r\left(f^{(m)}, 1/n\right)_{p(\cdot),\omega}.$$

Theorem 4 *If* $f \in W^{p(\cdot)}_{\omega,r}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega \in A_{p(\cdot)}(\mathbb{T})$ *and* $r \in \mathbb{R}^+$, *then* $\Omega_r(f,\delta)_{p(\cdot),\omega} \leq c(p,r)\delta^r \left\|f^{(r)}\right\|_{p(\cdot),\omega}$.

Theorem 5 *Let* $f \in L^{p(\cdot)}_{\omega}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$. *If* $\sum\limits_{\nu=1}^{\infty} \nu^{k-1}E_\nu(f)_{p(\cdot),\omega} < \infty$ *for some* $k \in \mathbb{R}^+$, *then* $f \in W^{p(\cdot)}_{\omega,k}(\mathbb{T})$ *and for* *every* $n \in \mathbb{N}$

$$E_n\left(f^{(k)}\right)_{p(\cdot),\omega} \leq c(p,k)\left\{n^k E_n\left(f\right)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} \nu^{k-1} E_\nu\left(f\right)_{p(\cdot),\omega}\right\}.$$

Remark Note that in the nonweighted spaces $L^{p(\cdot)}(\mathbb{T})$ the results discussed in this paper in term of the integer order modulus of smoothness for the more general exponents $p(\cdot)$, namely when $p(\cdot) \in \mathcal{P}(\mathbb{T})$ were announced in [3].

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Inverse Problem For 2D Heat Equation

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Abstract

We consider an inverse problem for two-dimensional heat equation containing two unknown coefficients: one of them depends on time variable and one space variable, another depends on time variable and the second space variable. Using Green function and applying Schauder fixed point theorem, we establish conditions for existence of a classical solution of the problem. The uniqueness of solution is also established.

Keywords: inverse problem, two dimensional heat equation.

1 Introduction

Coefficient inverse problems for parabolic equations in 1D case are well studied . The particularity of this case consists of the fact that unknown coefficients may depend only on one variable. When one wants to pass to 2D case, a new possibility turns out: unknown coefficients can depend on time and space variables simultaneously. Such a kind of problem is considered in this paper: an inverse problem for 2D heat equation with two unknown coefficients depending on time and one of the space variables is studied.

2 Statement of the problem

Consider an inverse problem for finding coefficients $a(y,t), b(x,t)$ and function $u(x,y,t)$ from equation

$$u_t = a(y,t)u_{xx} + b(x,t)u_{yy} + f(x,y,t), \quad (x,y,t) \in Q_T, \tag{1}$$

initial condition

$$u(x, y, 0) = \varphi(x, y), \quad (x, y) \in \overline{D}, \tag{2}$$

boundary and overdetermination conditions

$$u(0, y, t) = \mu_{11}(y, t),\ u(h, y, t) = \mu_{12}(y, t),\ (y, t) \in [0, l] \times [0, T], \tag{3}$$
$$u(x, 0, t) = \mu_{21}(x, t),\ u(x, l, t) = \mu_{22}(x, t),\ (x, t) \in [0, h] \times [0, T], \tag{4}$$
$$a(y, t)u_x(0, y, t) = \mu_{31}(y, t), \quad (y, t) \in [0, l] \times [0, T], \tag{5}$$
$$b(x, t)u_y(x, 0, t) = \mu_{32}(x, t), \quad (x, t) \in [0, h] \times [0, T], \tag{6}$$

where $D := \{(x, y) : 0 < x < h, 0 < y < l\}, Q_T := D \times (0, T)$.

A triple of functions $(a(y, t), b(x, t), u(x, y, t)$ will be called a solution of the problem (1)-(6) if it belongs to the space $(C^{1,0}([0, l] \times [0, T]) \times C^{1,0}([0, h] \times [0, T]) \times C^{2,1}(\overline{Q}_T))$ and verifies (1)-(6). Moreover, $a(y, t) > 0, (y, t) \in [0, l] \times [0, T], b(x, t) > 0, (x, t) \in [0, h] \times [0, T]$.

Suppose that the following assumptions hold:

(1) $\varphi \in C^2(\overline{D}), \mu_{1i} \in C^{2,1}([0, l] \times [0, T]), \mu_{2i} \in C^{2,1}([0, h] \times [0, T]), i \in \{1, 2\}, \mu_{31} \in C^{1,0}([0, l] \times [0, T]), \mu_{3,2} \in C^{2,1}([0, h] \times [0, T]), f \in C^{1,0}(\overline{Q}_T)$;

(2) $\varphi_x(x, y) > 0, \varphi_y(x, y) > 0, (x, y) \in \overline{D}; \mu_{1i_y}(y, t) > 0, \mu_{11_t}(y, t) - f(0, y, t) - b(0, t)\mu_{11_{yy}}(y, t) \leq 0, \mu_{12_t}(y, t) - f(h, y, t) - b(h, t)\mu_{12_{yy}}(y, t) \geq 0, mu_{31}(y, t) > 0, (y, t) \in [0, l] \times [0, T]; \mu_{2i_x}(x, t) > 0, i \in \{1, 2\}, \mu_{21_t}(y, t) - f(x, 0, t) - a(0, t)\mu_{21_{xx}}(x, t) \leq 0, \mu_{22_t}(y, t) - f(x, l, t) - a(l, t)\mu_{22_{xx}}(x, t) \geq 0, mu_{32}(x, t) > 0, (x, t) \in [0, h] \times [0, T], ; f_x(x, y, t) \geq 0, f_y(x, y, t) \geq 0, (x, y, t) \in \overline{Q}_T$;

(3) consistency conditions of the zero order .

3 Existence of solution

Theorem 1. *Suppose that the assumptions* **(1)**-**(3)** *are fulfilled. Then such a number* $T_1 \in (0, T]$ *may be indicated that there exists a solution of the problem (1)-(6) defined for* $(x, y) \in \overline{D}, t \in [0, T_1]$.

To prove the theorem, we find the solution of the problem (1)-(4) using the Green function:

$$u(x,y,t) = \iint\limits_D G_{11}(x,y,t,\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta + \int\limits_0^t \int\limits_0^l G_{11_\xi}(x,y,t,0,\eta,\tau)$$

$$\times\, a(\eta,\tau)\mu_{11}(\eta,\tau)d\eta d\tau - \int\limits_0^t \int\limits_0^l G_{11_\xi}(x,y,t,h,\eta,\tau)a(\eta,\tau)\mu_{12}(\eta,\tau)d\eta d\tau$$

$$+ \int\limits_0^t \int\limits_0^h G_{11_\eta}(x,y,t,\xi,0,\tau)b(\xi,\tau)\mu_{21}(\xi,\tau)d\xi d\tau - \int\limits_0^t \int\limits_0^h G_{11_\eta}(x,y,t,\xi,l,\tau)$$

$$\times\, b(\xi,\tau)\mu_{22}(\xi,\tau)d\xi d\tau + \int\limits_0^t \iint\limits_D G_{11}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau,$$

$$(x,y,t) \in \overline{Q}_T. \tag{7}$$

Calculating from (7) the derivatives u_x, u_y and substituting them into (5), (6), we obtain a system of equations with respect to $a(y,t), b(x,t)$. We apply the Schauder fixed-point theorem to this system and we obtain the existence of its solution. Then we find the function $u(x,y,t)$ as a solution to the corresponding direct problem.

4 Uniqueness of solution

Theorem 2. *Suppose that* $\mu_{31}(y,t) \neq 0$, $(y,t) \in [0,l]\times[0,T]$, $\mu_{32}(x,t) \neq 0$, $(x,t) \in [0,h]\times[0,T]$. *Then the solution of the problem (1)-(6) is unique.*

The proof is organized by the usual way: we suppose existence of two solutions $(a_k(y,t), b_k(x,t), u_k(x,y,t)), k \in \{1,2\}$ for the problem (1)-(6) and for their difference we obtain a homogeneous inverse problem. With the aid of the Green function, we reduce this problem to a system of homogeneous Volterra integral equations with respect to

$a_1(y,t) - a_2(y,t)$ and $b_1(x,t) - b_2(x,t)$ that has only a trivial solution. Then we use the uniqueness of solution of the direct problem and obtain $u_1(x,y,t) \equiv u_2(x,y,t)$. The proof is complete.

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Uncontrollable Distortions in Inverse Problems for Dynamical Systems

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Abstract

The influence of errors in the initial conditions on the solution of the inverse problem for a dynamical system is studied. It is shown that these errors lead to uncontrolled distortions of the solution of the inverse problem. A method is proposed for special filtering of initial data which allows to exclude the uncontrolled distortions.

Keywords: dynamical systems, inverse problem, uncontrolled distortions, filtering.

1 Introduction

The inaccuracy is inevitable in experimental measuring of physical values. It consist of inaccuracy of measuring instruments, noise value and inaccuracy of visual means. The value of this inaccuracy can be evaluated by technical indicators of measuring instruments. They do not exceed 5-10 percent as a rule.

The experimental measuring are chosen as initial data for the following calculations with the use of mathematical models in many practical important problems. For example, the inverse problems for evolution processes [1], the control problems with the use of experimental data [2] belong to this class.

2 Statement of the Problem

Let us consider the certain dynamic system the motion of which is describing by the equation

$$\dot{X} = CX(t) + BZ(t), \tag{1}$$

where initial conditions

$$X(0) = X^0, \tag{2}$$

where $Z(t)$ is the vector function of external impacts, $X(t)$ is the vector-function of state variables, C is matrix of system, B is matrix of control.

The vector function $Z(t)$ is given in direct problems. The matrices C and B are also given. The vector function $X(t)$ is an unknown function. The initial conditions (2) have been given. The solution of system (1) can be presented in the form

$$X(t) = F(X^0, Z(t)). \tag{3}$$

If we consider the inverse problems, for example, when the vector function $Z(t)$ is searched, then we use the vector function $X(t)$, values X^0 and matrixes C, B, as initial data. If we have all components of $X(t)$ then we have the values X^0. But as a rule in practice we can't measure all components of $X(t)$. One or two components of vector function $X(t)$ are measured usually, for example, only the first component $x_1(t)$. Then it is necessary to have the values $\dot{x}_1(t), \ddot{x}_1(t)$... for the search of vector function $Z(t)$. But the inaccuracy of $\dot{x}_1(t), \ddot{x}_1(t)$... cannot be evaluated in principle as the function $x_1(t)$ was obtained by experimental way with error. This inaccuracy equals infinity in general case. It leads to approximate solution which will be equal zero if the regularization method was used [3]. The indicated inaccuracy was called the uncontrollable inaccuracy [4].

3 The Filtration of Initial Data

In work [4] it was shown that uncontrollable inaccuracy lead to an additional uncontrollable distortions of the desired solution of the form

$$c_0 + c_1 t + c_2 t^2 + ... + k_1 \delta_+(t) + k_2 \dot{\delta}_+(t) + ... \tag{4}$$

where c_0, c_1, k_1, k_2 are constants, $\delta_+(t)$ is symmetrical delta function.

Let us consider the inverse problem for the dynamical system (1) as the solution of equation

$$Az = u, \tag{5}$$

where z is searched function, u is given function, A is given compact operator.

The following method of influence removal of uncontrollable inaccuracy on result of inverse problem solution is suggested: the items which determine the uncontrollable values of initial conditions are excluded from function u in equation (5) by means of special filtration. For example, for the inverse problem of astrodynamics [5] in the equation (5) the items c_0, c_1 are excluded by us from function u as very these items determine the uncontrolled distortions. Further we used the properties of Legander's polynomials. Let us define the values of c_0, c_1 from expressions $c_0 = \tilde{c}_0 - 0.5\tilde{c}_2, c_1 = \tilde{c}_1 - 1.5\tilde{c}_3$, where $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ are the coefficients of Fourier of function u on Legander's polynomials. Then in equation (5) we use the function $\hat{u} = u - c_0 - c_1 t$ instead of function u.

4 Conclusion

An algorithm for eliminating an uncontrolled error in solving an inverse problem for a dynamical system based on a special filtering of the initial data is proposed.

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Invariant conditions of stability of unperturbed motion for differential systems with quadratic nonlinearities in the critical case

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Abstract

The center-affine invariant conditions of stability of unperturbed motion governed by differential systems in the plane with quadratic nonlinearities in the critical case were determined.

Keywords: Differential systems, stability of unperturbed motions, center-affine comitants and invariants, Sibirsky graded algebras.

1 Introduction

Differential systems with polynomial nonlinearities play an important role in practical problems. Among them, the more spread are the Lyapunov critical systems, i.e. the systems with one root of the characteristic equation equal to zero and the others roots with negative real parts. In this paper the systems with quadratic nonlinearities of the Lyapunov form are studied.

2 Stability of unperturbed motion

We examine the differential system with quadratic nonlinearities

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = 1, 2), \tag{1}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices in which the total convolution is done.

The tensorial forms of generators of the Sibirsky algebras [1] of the system (1) will be written [2]:

$$I_1 = a^\alpha_\alpha,\ \ I_2 = a^\alpha_\beta a^\beta_\alpha,\ \ I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta}\varepsilon^{pq},\ \ K_1 = a^\alpha_{\alpha\beta}x^\beta,\ \ K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq},$$

$$K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma}x^\gamma,\ \ K_4 = a^\alpha_\gamma a^\beta_{\alpha\beta}x^\gamma,\ \ K_5 = a^p_{\alpha\beta}x^\alpha x^\beta x^q \varepsilon_{pq},\ \ K_7 = a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta,$$

$$K_8 = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta}x^\delta,\ \ K_{11} = a^p_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma x^q \varepsilon_{pq},\ \ K_{12} = a^\alpha_\beta a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}x^\delta x^\mu, \tag{2}$$

$$K_{13} = a^\alpha_\gamma a^\beta_{\alpha\beta}a^\gamma_{\delta\mu}x^\delta x^\mu,$$

where $\varepsilon^{pq}(\varepsilon_{pq})$ is the unit bivector with coordinates.

It is easy to show that if the invariant conditions are satisfied

$$I_1^2 - I_2 = 0,\ \ I_1 < 0, \tag{3}$$

then system (1), by a center-affine transformation, can be brought to the following critical system of the Lyapunov form

$$\frac{dx^1}{dt} = a^1_{\alpha\beta}x^\alpha x^\beta,\ \ \frac{dx^2}{dt} = a^2_\alpha x^\alpha + a^2_{\alpha\beta}x^\alpha x^\beta\ \ (\alpha,\beta = 1,2). \tag{4}$$

Remark 1. In this paper the Lyapunov Theorem on stability of unperturbed motion [3, §32] will be called the Lyapunov Theorem.

Let us introduce the following notations

$$P = (a^2_2)^2 a^1_{11} - 2a^2_1 a^2_2 a^1_{12} + (a^2_1)^2 a^1_{22},\ Q = (a^2_2)^2 a^2_{11} - 2a^2_1 a^2_2 a^2_{12} + (a^2_1)^2 a^2_{22},$$

$$R = (a^2_2)^2 a^1_{11} - (a^2_1)^2 a^1_{22},\ S = a^2_1 a^1_{22} - a^2_2 a^1_{12}\quad (I_1 = a^2_2 < 0). \tag{5}$$

Taking into account (5) and the Lyapunov Theorem on stability of unperturbed motion in system (4) we have the following lemma.

Lemma 1. *The stability of unperturbed motion in system* (4) *is described by one of the following six possible cases:*

I. $P \neq 0$, *then the unperturbed motion is unstable;*

II. $P = 0, QS > 0$, *then the unperturbed motion is unstable;*

III. $P = 0, QS < 0$, *then the unperturbed motion is stable;*

IV. $R = S = 0,\ a^1_{22}Q \neq 0$, *then the unperturbed motion is unstable;*

V. $P = Q = 0$, *then the unperturbed motion is stable;*

VI. $a^1_{11} = a^1_{12} = a^1_{22} = 0$, *then the unperturbed motion is stable.*

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motion, moreover in Case (iii) it is also asymptotic stable [4]. The expressions P, Q, R, S *are given in* (5).

Later on, we make use of the following expressions of the invariants and comitants of system (1) given in (2):

$$E_1 = I_1^2K_1 - I_1(K_3 + K_4) + K_8,$$

$$E_2 = I_1^3(K_1^2 - K_7) + 2I_1^2(K_1K_4 - 2K_1K_3 - K_{13}) + 2I_1(I_5K_2 + 2K_3^2 - K_4^2) +$$

$$+4K_8(K_4 - K_3) + 2I_2K_{12}, \ \ E_3 = I_2K_1 + I_1(K_4 - K_3) - K_8,$$

$$E_4 = I_1(K_{11} - K_1K_2) + K_2(K_4 - K_3), \ \ E_5 = K_{11} - I_1K_5. \quad (6)$$

Theorem 1. *Let for differential system of the perturbed motion* (1) *the invariant conditions* (3) *are satisfied. Then the stability of the unperturbed motion in system (1) is described by one of the following six possible cases:*

I. $E_1 \not\equiv 0$, *then the unperturbed motion is unstable;*

II. $E_1 \equiv 0,\ \ E_2 > 0$, *then the unperturbed motion is unstable;*

III. $E_1 \equiv 0,\ \ E_2 < 0$, *then the unperturbed motion is stable;*

IV. $E_3 \equiv 0, E_4E_5 \not\equiv 0$, *then the unperturbed motion is unstable;*

V. $E_4 \equiv 0$, *then the unperturbed motion is stable;*

VI. $E_5 \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motion, and moreover in Case III it is also asymptotic stable. The expressions E_i $(i = \overline{1,5})$ *are given in* (6).

Remark 2. The extended conditions for Lyapunov's example [3, §32] are obtained from Theorem 1.

3 Conclusion

In this paper the differential system given in Example 2 [3, §32] was investigated by means of comitants and invariants of the Sibirsky algebras of differential system with quadratic nonlinearities.

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On the Riemann boundary value problem in the case of a piecewise Lyapunov contour

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Abstract

The generalized Riemann boundary value-problem is investigated in the case of a piecewise Lyapunov contour. It is proved that the conditions for normal solvability depend on the coefficients of the problem, as well as on the presence of corner points on the contour of integration.

Keywords: Noetherian operator, Lyapunov contour, symbol, Riemann boundary value problem.

1 Introduction

Let Γ be a closed, oriented, piecewise Lyapunov contour which divides the complex plan into an interior domain D^+ and an exterior domain D^-. We denote by $L_p(\Gamma,\rho)$ the space L_p on Γ with weight $\rho(t) = \prod_{k=1}^{n} |t-t_k|^{\beta_k}$, where $t_1,\dots,t_n$ are distinct points of the curve Γ and $\beta_1,\dots,\beta_n$ are arbitrary real numbers satisfying the relations $-1<\beta_k< p-1$. In the space $L_p(\Gamma,\rho)$, over the field of real numbers, we consider the bounded linear operator

$$A = aP + bQ + (cP + dQ)V, \tag{1}$$

where a, b, c, d are continuous functions on Γ, $(V\phi)(t) = \bar{\phi}(t)$, and

$$(P\phi)(t) = \frac{1}{2}\phi(t) + \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t}d\tau,\ (Q\phi) = \phi(t) - (P\phi)(t).$$

In the case when Γ is a Lyapunov contour the operator A and the Carleman integral equations with a shift and complex conjugate unknowns have been considered by many authors. We mention only [1], where a detailed bibliography can be found.

In constructing the Noether theory of the operator A a basic role is played by the fact that if at each point of the contour Γ a Lyapunov condition is satisfied, then the operator $VSV + S$ $(S = P + Q)$ is completely continuous in space $L_p(\Gamma, \rho)$ (see [1]). In this case A is Noetherian [1] if and only if the operator

$$A_V = \begin{Vmatrix} a & c \\ \bar{d} & \bar{b} \end{Vmatrix} P + \begin{Vmatrix} b & d \\ \bar{c} & \bar{a} \end{Vmatrix} Q \tag{2}$$

possesses the same property in the space $L_p^2(\Gamma, \rho) = L_p(\Gamma, \rho) \times L_p(\Gamma, \rho)$. The situation is otherwise if the contour Γ has corner points. It turns out that in this case the operator $VSV + S$ is not completely continuous in $L_p(\Gamma, \rho)$, and if A is Noetherian, so is A_V, but the converse assertion is not true. These facts disclose the essential difference between the piecewise Lyapunov contour and a Lyapunov contour.

In this note we construct the symbol of the operator (1) in the form of a matrix-valued function of the variable order. The non-degeneracy of the symbol is a necessary and sufficient condition for the operator A to be Noetherian in $L_p(\Gamma, \rho)$. Analogous results are obtained for the generalized Riemann boundary value problem.

2 Defining the symbol

Let $t_1, \ldots, t_n$ be all the corner points of the contour Γ, where $t_1 \prec t_2 \prec \cdots \prec t_n$ and the relation $t_k \prec t_{k+1}$ means that the point t_k precedes t_{k+1} on the oriented contour Γ. We denote by $\tilde{\alpha}_k,\ k = 1, 2, \ldots, n$, the non-negative angle, $0 \le \tilde{\alpha}_k \le 2\pi$, by which an infinitely small vector $t_k Z$ rotates when the point Z to the left of Γ and turning about t_k passes from the portion $t_k t_{k+1},\ k = 1, 2, \ldots, n,\ t_{n+1} = t_1$ of the contour Γ to the portion $t_{k-1} t_k$ $(t_0 = t_1)$. We denote by α_k $(= \alpha(t_k))$ the quantity $\alpha_k = min(\tilde{\alpha}_k,\ 2\pi - \tilde{\alpha}_k)$. In this paper we assume that $\alpha_k \neq 0$.

We now define the symbol $A(t,\xi)$, $t\in\Gamma$, $-\infty\le\xi\le\infty$ of the operator A acting in $L_p(\Gamma,\rho)$. To this end we first define the symbol of the operators aI, P and Q. The symbol aI , $a\in C(\Gamma)$, is the matrix-valued function $a(t,\xi)$, $t\in\Gamma$, $-\infty\le\xi\le\infty$ of variable order defined by the following equalities:

$$a\,(t,\xi)=\begin{cases} diag(a(t),\ \overline{a\,(t)})\ \ for\ t\neq t_k, \\ diag(a(t_k),\ \overline{a\,(t_k)},\ a(t_k),\ \overline{a\,(t_k)}),\end{cases}$$

where $diag(x_1,\,x_2,\ldots,x_s)$ is the diagonal matrix of order s with the elements $x_1,\,x_2,\ldots,x_s$ on the diagonal.

The symbol $P(t,\xi)$ of an operator P is the following matrix-valued function

$$P(t,\xi)=\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}\ \text{for}\ t\in\Gamma\backslash\left\{t_1,t_2,\ldots,t_n\right\},$$

$$P(t,\xi)=\frac{1}{z_k^{2\pi}-1}\begin{Vmatrix} z_k^{2\pi} & 0 & -z_k^{\alpha_k} & 0 \\ 0 & -1 & 0 & z_k^{2\pi-\alpha_k} \\ z_k^{2\pi-\alpha_k} & 0 & -1 & 0 \\ 0 & -z_k^{\alpha_k} & 0 & z_k^{2\pi} \end{Vmatrix},$$

where $z_k=\exp(\xi+i\frac{1+\beta_k}{p})$. We define the symbol $Q(t,\xi)$ of the operator Q by the formula $Q(t,\xi)=E(t)-P(t,\xi)$, where $E(t)$ is the identity matrix of second order for $t\neq t_k$ and of fourth order for $t=t_k,\ k=1,2,\ldots,n$.

The symbol $V\,(t,\xi)$ of the operator V is defined by the matrices

$$V(t,\xi)=\begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix},\ \text{for}\,t\in\Gamma\backslash\left\{t_1,\ldots,t_n\right\},\ V(t_k,\xi)=\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}.$$

If the operator A has the form (1), we define its symbol $A(t,\xi)$ to be

$$A\,(t,\xi)=a\,(t,\xi)\,P\,(t,\xi)+b\,(t,\xi)\,Q\,(t,\xi)+$$

$$+ \left[c\left(t,\xi\right)P\left(t,\xi\right) + d\left(t,\xi\right)Q\left(t,\xi\right) \right] V\left(t,\xi\right).$$

Theorem 1. *The operator $A = aP + bQ + (cP + dQ)V$ is Noetherian in the space $L_p(\Gamma,\rho)$ if and only if the following condition is satisfied*

$$\det A(t,\xi) \neq 0,\ t \in \Gamma,\ -\infty \leq \xi \leq \infty.$$

Corollary 1. *If the operator A is Noetherian, then the corresponding operator A_V defined by* (2) *is also Noetherian. The converse is not true, in general.*

The properties of operators of local type [2] and some results from [3] and [4] concerning singular operators with a shift along piecewise Lyapunov curves are used in the proof of Theorem 1.

Theorem 2. *The operator*

$$(VSV+S)\phi = \frac{1}{\pi i}\int_\Gamma \frac{\phi\left(\tau\right)}{\bar{\tau}-\bar{t}}\overline{d\tau} + \frac{1}{\pi i}\int_\Gamma \frac{\phi\left(\tau\right)}{\tau - t} d\tau$$

is completely continuous in space $L_p(\Gamma,\rho)$ if and only if Γ is a Lyapunov contour.

The sufficient part of this assertion was proved in [1,5]. We prove here the necessity. Suppose that $VSV + S$ is completely continuous, then the operator $R_\lambda = VSV + S - \lambda I$ is Noetherian for all $\lambda \in \mathbb{C}\setminus\{0\}$. Hence, by Theorem 1, $\det R_\lambda(t_k,\xi) \neq 0$ for all $k = 1, 2, \ldots, n$ and $-\infty \leq \xi \leq \infty$. From this we find that $\frac{z_k^{2\pi-\alpha_k} - z_k^{\alpha_k}}{z_k^{2\pi}-1} \equiv 0$, where $z_k = \exp(\xi + i\frac{1+\beta_k}{p})$.
The latter is possible only for $\alpha_k = \pi$. This means that Γ is a Lyapunov contour. The proof of the theorem is complete.

In contrast to singular operators don't containing the operator V (i.e., $A = aP + bQ$), the condition for the operator A be Noetherian essentially depends on the contour. For example, the operator $A = (1+\sqrt{2})P + (1-\sqrt{2})Q + V$ is Noetherian in all spaces $L_p(\Gamma,\rho)$, if Γ is a Lyapunov contour and is not Noetherian in $L_2(\Gamma)$, if Γ has at least one corner point with angle $\pi/2$. This follows immediately from Theorem 1.

3 Noetherian criteria

In conclusion we consider the generalized Riemann boundary value problem: find analytic functions $\Phi^{+}(z)$ and $\Phi^{-}(z)$ which can be represented by the Cauchy integral in $D^{+}$and D^{-}, with limit values on Γ which belong to $L_p(\Gamma,\rho)$, $1<p<\infty$, and satisfy the boundary condition

$$\Phi^{+}(t)=a(t)\Phi^{-}(t)+b(t)\overline{\Phi^{-}(t)}+c(t), \tag{3}$$

where $a(t)$ and $b(t)$ are known continuous functions on Γ and $c(t)\in L_p(\Gamma,\rho)$.

The Noether theory to the problem (3) in the case of a Lyapunov contour is constructed in [1] and [6]. In particular, in these papers it was established that the inequality $|a(t)|>0$ for all $t\in\Gamma$ is a necessary and sufficient condition for the problem to be Noetherian. In the case of a piecewise Lyapunov contour we have the following theorem.

Theorem 3. *The following conditions are necessary and sufficient for the problem* (3) *be Noetherian:*

1. $|a(t)|>0,\ t\in\Gamma$;
2. $|a(t_k)|^2-\dfrac{z_k^{2\pi-\alpha_k}-z_k^{\alpha_k}}{z_k^{2\pi}-1}|b(t_k)|^2\neq 0$, for all $k=1,2\ldots,n$,

where

$$z_k=\exp(\xi+i\frac{1+\beta_k}{p}),\ -\infty\leq\xi\leq\infty.$$

Thus, in the case of a piecewise Lyapunov contour, the Noetherian property of the problem (3) depends not only by the coefficient $a(t)$, as in the case of a Lyapunov contour, but also on $b(t)$.

4 Conclusion

The Noetherian criteria for the Riemann boundary-value problems were obtained on piecewise Lyapunov curves applying the Plemelj-Sohotsky formulas in combination with the symbol of singular integral equations. The proposed method can be used in the study of boundary-value problems with discontinuous coefficients when the integration contour consists of a finite number of closed curves without common points.

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On periodic solutions of the four-dimensional differential system of Lyapunov-Darboux type with quadratic nonlinearities

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Abstract

For the four-dimensional differential system of Lyapunov-Darboux type with quadratic nonlinearities, we have found a holomorphic integral of Lyapunov type. Using this integral and the Lyapunov theorem, we have obtained centro-affine invariant conditions for stability of unperturbed periodic motion.

Keywords: Differential system, center-affine comitant, stability of unperturbed motion.

1 Introduction

The differential systems with polynomial nonlinearities are important in various applied problems. One of the interesting cases is differential systems, which characteristic equations have purely imaginary roots. In this paper we consider a four-dimensional differential system, which characteristic equation has two simple purely imaginary roots and the other two roots have real negative parts.

2 Stability of unperturbed periodic motions of Lyapunov-Darboux type differential system with quadratic nonlinearities

Let the characteristic equation of the differential system

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + X^j \quad x^1, x^2, ..., x^{n+2}) \quad (j, \alpha = \overline{1, n+2}) \tag{1}$$

has two purely imaginary simple roots $\lambda\sqrt{-1}$ and $-\lambda\sqrt{-1}$, where X^j are holomorphic functions of x^j $(j = \overline{1,4})$.

The systems (1) with two purely imaginary roots and n roots with negative real part of the characteristic equation will be called *systems of Lyapunov type*.

Consider the system of differential equations

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = \overline{1,4}), \tag{2}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices, in which the complete convolution is made and the group of center-affine transformations $GL(4, \mathbb{R})$.

The following center-affine invariant polynomials of the system (2) are known from [3]:

$$I_{1,4} = a^\alpha_\alpha,\ I_{2,4} = a^\alpha_\beta a^\beta_\alpha,\ I_{3,4} = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta,\ I_{4,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta a^\delta_\gamma,\ P_{1,4} = a^\alpha_{\alpha\beta} x^\beta,$$

$$P_{2,4} = a^\alpha_\beta a^\beta_{\alpha\gamma} x^\gamma,\ P_{3,4} = a^\alpha_\gamma a^\beta_\alpha a^\gamma_{\beta\delta} x^\delta,\ P_{4,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta a^\delta_{\gamma\mu} x^\mu,$$

$$S_{0,4} = u_\alpha x^\alpha,\ S_{1,4} = a^\alpha_\beta x^\beta u_\alpha,\ S_{2,4} = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta,\ S_{3,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta x^\delta u_\gamma,$$

$$\overline{R}_{6,4} = a^\alpha_p a^\beta_q a^\gamma_\beta a^\delta_r a^\mu_\delta a^\nu_\mu u_s u_\alpha u_\gamma u_\nu \varepsilon^{pqrs},\ \overline{R}_{6,4} = det\left(\frac{\partial S_{i-1,4}}{\partial x^j}\right)_{i,j=\overline{1,4}},$$

$$K_{6,4} = a^\alpha_\theta a^\beta_\gamma a^\gamma_\varphi a^\delta_\mu a^\mu_\nu a^\nu_\psi x^\theta x^\varphi x^\psi x^\tau \varepsilon_{\alpha\beta\delta\tau},\ \widetilde{K}_{1,4} = a^\alpha_{\beta\gamma} x^\beta x^\gamma y^\delta z^\mu \varepsilon_{\alpha\gamma\delta\mu}, \tag{3}$$

where vectors $y = (y^1, y^2, y^3, y^4)$ and $z = (z^1, z^2, z^3, z^4)$ are cogradient with vector of phase variables $x = (x^1, x^2, x^3, x^4)$, and vector

$u = \ (u_1, u_2, u_3, u_4)$ is covariant with vector x [2]. Polynomials $I_{i,4}$ $(i = \overline{1,4})$ are invariants, $P_{j,4}$ $(j = \overline{1,4})$ and $K_{6,4}$ are comitants, $S_{j,4}$ $(j = \overline{0,3})$ are mixed comitants, $\overline{R}_{6,4}$ is contravariant, and $\widetilde{K}_{1,4}$ is comitant of cogradient vectors x, y, z [2].

Lemma 1 [3]. *If $\widetilde{K}_{1,4}$ from* (3) *is identically equal to zero* $(\widetilde{K}_{1,4} \equiv 0)$ *then the system* (2) *takes the form*

$$\dot{x}^j = a^j_\alpha x^\alpha + 2x^j \ \ a^1_{1\alpha} x^\alpha) \quad (j, \alpha = \overline{1,4}). \tag{4}$$

The system (4) is called *four-dimensional differential system of Darboux type.*

Remark 1. *For any centro-affine transformation of the system* (4)*, its quadratic part retains its form changing only the variables and coefficients. This follows from the fact that the identity $\widetilde{K}_{1,4} \equiv 0$ is preserved under any centro-affine transformation.*

Lemma 2 [3]. *If $\overline{R}_{6,4} \not\equiv 0$, then by the centro-affine transformation*

$$\overline{x}^1 = S_{0,4}, \quad \overline{x}^2 = S_{1,4}, \quad \overline{x}^3 = S_{2,4}, \quad \overline{x}^4 = S_{3,4}, \tag{5}$$

the system (4) *can be brought to the following form:*

$$\dot{x}^1 = x^2 + 2x^1 \left(a^1_{1\alpha} x^\alpha\right), \ \dot{x}^2 = x^3 + 2x^2 \left(a^1_{1\alpha} x^\alpha\right), \ \dot{x}^3 = x^4 + 2x^3 \ \ a^1_{1\alpha} x^\alpha\right),$$

$$\dot{x}^4 = -L_{4,4} x^1 - L_{3,4} x^2 - L_{2,4} x^3 - L_{1,4} x^4 + 2x^4 \ \ a^1_{1\alpha} x^\alpha\right), \tag{6}$$

where

$$L_{1,4} = -I_{1,4}, \quad L_{2,4} = \frac{1}{2}\,(I^2_{1,4} - I_{2,4}), \quad L_{3,4} = \frac{1}{6}\,(3I_{1,4}I_{2,4} - 2I_{3,4} - I^3_{1,4}),$$

$$L_{4,4} = \frac{1}{24}\,(8I_{1,4}I_{3,4} - 6I_{4,4} - 6I^2_{1,4}I_{2,4} + 3I^2_{2,4} + I^4_{1,4}), \tag{7}$$

$I_{k,4}$ $(k = \overline{1,4})$ *are from* (3).

Remark 2. *The characteristic equation of the system* (6) *has the form*

$$\rho^4 + L_{1,4}\,\rho^3 + L_{2,4}\,\rho^2 + L_{3,4}\,\rho + L_{4,4} = 0, \tag{8}$$

where $L_{i,4}$ $(i=\overline{1,4})$ *are from* (7).

Lemma 3. *The characteristic equation* (8) *has two simple purely imaginary roots* $\lambda\sqrt{-1}$ *and* $-\lambda\sqrt{-1}$ *and two other roots with real negative part if and only if*

$$L_{1,4}>0,\quad L_{2,4}>0,\quad L_{1,4}L_{2,4}-L_{3,4}>0,$$

$$L_{1,4}^2L_{4,4}+L_{3,4}^2-L_{1,4}L_{2,4}L_{3,4}=0, \tag{9}$$

where $L_{i,4}$ $(i=\overline{1,4})$ *are from* (7).

Theorem 1. *If* $\widetilde{K}_{1,4}\equiv 0$ *and* $\overline{R}_{6,4}\not\equiv 0$*, then under conditions* (9)*, using centro-affine transformation the system* (2) *can be brought to the following form* ($x=x^1$, $y=x^2$, $z=x^3$, $u=x^4$)*:*

$$\dot{x}=-\lambda\, y+2x\psi\equiv P,\quad \dot{y}=\lambda\, x+2y\psi\equiv Q,\quad \dot{z}=u+2z\psi\equiv R,$$

$$\dot{u}=y+(\lambda^2+c)\, z+d\, u+2u\psi\equiv S, \tag{10}$$

where $\lambda=\pm\sqrt{\frac{L_{3,4}}{L_{1,4}}}$ $(L_{1,4}L_{3,4}>0)$, $c=-L_{2,4}$, $d=-L_{1,4}$, $L_{i,4}$ *are from* (7), $\psi=Ax+By+Cz+Du$ *with* A, B, C, D *real constants.*

The system (10) will be called *differential system of Lyapunov-Darboux type.*

Theorem 2. *The functions*

$$\zeta_1=x^2+y^2,\quad \zeta_2=\lambda^3+c\lambda-2(Bc-C+B\lambda^2)\,x+2A(c+\lambda^2)\,y+$$

$$+2\lambda(-Cd+cD+D\lambda^2)\,z+2C\lambda\, u,\quad \zeta_3=\lambda^2\,x^2+d\lambda\, xy+cd\lambda\, xz+$$

$$+\lambda(2c+d^2+4\lambda^2)\,xu-(c+\lambda^2)\,y^2-[2c^2+(6c+d^2)\lambda^2+4\lambda^4]\,yz-$$

$$-cd\,yu-[c^3+c(5c+d^2)\lambda^2+(8c+d^2)\lambda^4+4\lambda^6]\,z^2-$$

$$-[c^2+(4c+d^2)\lambda^2+4\lambda^4]\ \ dzu-u^2) \tag{11}$$

are particular integrals of the system (10), $F=\zeta_1\,\zeta_2^{-2}$ *is prime integral of Darboux type of the system* (10).

Proof. Denote by

$$\Lambda = P\,\frac{\partial}{\partial\,x} + Q\,\frac{\partial}{\partial\,y} + R\,\frac{\partial}{\partial\,z} + S\,\frac{\partial}{\partial\,u},$$

the operator of the system (10). By a direct calculation we obtain

$$\Lambda\,(\zeta_1) = 4\,\zeta_1\,\psi, \quad \Lambda\,(\zeta_2) = 2\,\zeta_2\,\psi, \quad \Lambda\,(\zeta_3) = \zeta_3\,(d + 4\psi),$$

$$\Lambda\left(\zeta_1^{\alpha}\,\zeta_2^{\beta}\right) = 2(2\alpha + \beta)\,\zeta_1^{\alpha}\,\zeta_2^{\beta}\,\psi,$$

where $\psi = Ax + By + Cz + Du$. The theorem is proved.

From [3] the following comitant of the system (2) is known:

$$\Phi_{4,4} = L_{4,4} - 2(4/5\,L_{3,4}P_{1,4} + L_{2,4}P_{2,4} + L_{1,4}P_{3,4} + P_{4,4}), \qquad (12)$$

where $P_{j,4}$ ($j = \overline{1,4}$) are from (3), $L_{i,4}$ ($i = \overline{1,4}$) are from (7).

Remark 3. *For the system* (10) *for $x = x^1$, $y = x^2$, $z = x^3$, $u = x^4$ we have $\Phi_{4,4} = -\lambda\zeta_2$, where ζ_2 is from* (11)*.*

Remark 4. *The prime integral $F = \zeta_1\,\zeta_2^{-2}$ of the system* (10) *with $\zeta_2 \not\equiv 0$ ($\Phi_{4,4} \not\equiv 0$) can be written as a holomorphic Lyapunov integral ([1], §40) $F = x^2 + y^2 + \widetilde{F}(x,y,z,u)$, where $\widetilde{F}(x,y,z,u)$ contains terms of degree at lest two in variables x, y, z, u.*

Using the Lyapunov theorem ([1], p.160), lemmas 1–3, theorems 1–2 and remarks 3–4, we obtain

Theorem 3. *Assume for the system* (2) *with $\widetilde{K}_{1,4} \equiv 0$ and $\overline{R}_{6,4} \not\equiv 0$ under centro-affine invariant conditions* (9)*, the comitant* (12) *is not identically zero. Then the system has a periodic solution containing an arbitrary constant, and varying this constant one can obtain a continuous sequence of periodic motions, which comprises the studied unperturbed motion. This motion is stable and any perturbed motion, sufficiently close to the unperturbed motion, will tend asymptotically to one of the periodic motions.*

3 Conclusion

Using of centro-affine invariants and comitants of the four-dimensional differential system with quadratic nonlinearities we obtain extension of the results stated in the Lyapunov theorem ([1], §40) concerning the stability of unperturbed periodic motion of the studied system.

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Large-time behavior of the difference of solutions of two evolution equation

Andrei Perjan, Galina Rusu

Abstract

In a real Hilbert space H we consider a linear self-adjoint positive definite operator $A : V = D(A) \subset H \to H$ and investigate the behavior of the difference $u - v$ of solutions to the problems

$$\begin{cases} u''(t) + u'(t) + Au(t) = f(t), \quad t > 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

$$\begin{cases} v'(t) + Av(t) = f(t), \quad t > 0, \\ v(0) = u_0, \end{cases}$$

where $u_0, u_1 \in H, f : [0, +\infty) \to H$.

Keywords: large-time behavior, abstract first order differential equation, abstract second order differential equation, a priori estimate.

Let H be a real Hilbert space endowed with the scalar product $(\cdot, \cdot)$ and the norm $|\cdot|$, and V be a real Hilbert space endowed with the norm $||\cdot||$, densely and continuously embedded in H i.e. there exists $\gamma_0 > 0$ such that

$$\gamma_0 \, |u| \leq ||u||, \quad \forall u \in V. \tag{\textbf{HV}}$$

Let $A : V \subset H \to H$ be a linear, self-adjoint and positive definite operator, i.e. there exists $\gamma > 0$ such that

$$(Au, u) \geq \gamma_1 \, ||u||^2, \quad \forall u \in V. \tag{\textbf{HA}}$$

Consider the following Cauchy problems:

$$\begin{cases} u''(t) + u'(t) + Au(t) = f(t), & t > 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{1}$$

$$\begin{cases} v'(t) + Av(t) = f(t), & t > 0, \\ v(0) = u_0, \end{cases} \tag{2}$$

where $u_0, u_1 \in H, f : [0, +\infty) \to H$.

We investigate the behavior of the difference $u - v$ of solutions to the problems (1) and (2) when $t \to +\infty$. The main result is established in Theorem 3. This result improves the results from [2] and [3] in the sense that we skip the condition of separability of the space H and take a nonzero right hand part therm.

For $k \in \mathbb{N}^\star$, $1 \leq p < +\infty$, $(a, b) \subset (-\infty, +\infty)$ and Banach space X we denote by $W^{k,p}(a, b; X)$ the Banach space of all vectorial distributions $u \in D'(a, b; X)$, $u^{(j)} \in L^p(a, b; X)$, $j = 0, 1, \dots, k$, endowed with the norm $\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)}\right)^{1/p}$. For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1, \infty]$, we define the Banach space $W^{k,p}_s(a, b; X) = \{f : (a, b) \to H; f^{(l)}(\cdot)e^{s\,t} \in L^p(a, b; X),\, l = 0, \dots, k\}$, with the norm $\|f\|_{W^{k,p}_s(a,b;X)} = \|f e^{s\,t}\|_{W^{k,p}(a,b;X)}$.

The results concerning the solvability of problems (1) and (2) are established in the following two theorems.

Theorem 1 [1]. *Let $t > 0$. Let us assume that the conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$, $u_1 \in H$ and $f \in W^{1,1}(0, t; H)$, then there exists the unique function $u \in W^{2,\infty}(0, t; H) \bigcap W^{1,\infty}(0, t; V)$, (strong solution) which satisfies the equation a.e. on $(0, t)$ and the initial conditions from* (1).

Theorem 2 [1]. *Let $t > 0$. Let us assume that the conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$ and $f \in W^{1,1}(0, t; H)$, then there exists the unique function $v \in W^{1,2}(0, t; V)$ which satisfies a. e. on $(0, t)$ the equation and the initial conditions from* (2).

Lemma 1. *Assume that conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$ and $f \in W^{1,1}(0, t; H)$, for every $t > 0$, then there exists the* *constant $C > 0$ such that for the strong solution v to the problem (2) the following estimate is valid:*

$$|v(t)| \leq C(\gamma) \left(|u_0| + \int_0^t e^{\gamma\tau}|f(\tau)|d\tau\right)e^{-\gamma t}, \forall t \geq 0, \ \gamma = \gamma_0\,\gamma_1.$$

Lemma 2. *Assume that conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$, $u_1 \in H$ and $f \in W^{1,1}(0,t;H)$, for every $t > 0$, then there exists the constant $C = C(\gamma) > 0$ and $\delta = \delta(\gamma) \in (0,1/2)$, such that for the strong solution u to the problem (1) the following estimate is valid:*

$$|u'(t)| + |A^{1/2}u(t)| \leq C\left(||u_0|| + |u_1| + \int_0^t e^{\delta\tau}|f(\tau)|d\tau\right)e^{-\delta\, t}, \forall t \geq 0.$$

In what follows, denote by

$$K(t,\tau) = \frac{1}{2\sqrt{\pi}}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau) - 2K_3(t,\tau)\Big),$$

$K_1(t,\tau) = \exp\left\{\frac{3t-2\tau}{4}\right\}\lambda\left(\frac{2t-\tau}{2\sqrt{t}}\right)$, $K_2(t,\tau) = \exp\left\{\frac{3t+6\tau}{4}\right\}\lambda\left(\frac{2t+\tau}{2\sqrt{t}}\right)$, $K_3(t,\tau) = \exp\left\{\tau\right\}\lambda\left(\frac{t+\tau}{2\sqrt{t}}\right)$, $\quad \lambda(s) = \int_s^\infty e^{-\eta^2}d\eta$.

The properties of kernel $K(t,\tau)$ are collected and proved in [4] and they allow to obtain the following lemma.

Lemma 3. *Assume that conditions* **(HV)** *and* **(HA)**. *If $u_0 \in V$, $u_1 \in H$ and $f \in W^{1,1}_\alpha(0,\infty;H)$, with some $\alpha > 0$, then the function*

$$w(t) = \int_0^\infty K(t,\tau)\,u(\tau)\,d\tau,$$

where u is the strong solution to the problem (2), verifies the system

$$\begin{cases} w'(t) + Aw(t) = F_0(t), \quad t > 0, \quad in \quad H, \\ w(0) = \displaystyle\int_0^\infty e^{-\tau}\,u(2\tau)\,d\tau, \end{cases}$$

$$F_0(t) = \frac{1}{\sqrt{\pi}}\left[2\exp\left\{\frac{3t}{4}\right\}\lambda\left(\sqrt{t}\right) - \lambda\left(\frac{1}{2}\sqrt{t}\right)\right]u_1 + \int_0^\infty K(t,\tau)\,f(\tau)\,d\tau.$$

The main result of the paper is presented in the following theorem and it follows from the Lemmas 1-3.

Theorem 3. *Assume that conditions* **(HV)** *and* **(HA)** *are fulfilled. If* $u_0 \in V$, $u_1 \in H$ *and* $f \in W_\alpha^{1,1}(0,\infty;H)$, *with some* $\alpha > 0$, *then there exists the constant* $C(\gamma,\alpha) > 0$ *and* $\delta(\gamma,\alpha) > 0$, *such that for the strong solutions* u *and* v *to the respective problems (1) and (2) the estimate*

$$|u(t) - v(t)| \leq C\Big(||u_0|| + |u_1| + ||f||_{W_\alpha^{1,1}(0,\infty;H)}\Big)e^{-\delta t}, \quad \forall t \geq 0,$$

is true.

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Boundary value problems for the Schrödinger equation with conditions at infinity

Oleksandr Pokutnyi

Abstract

The paper is devoted to investigation of boundary value problems for the evolution Schrödinger equation. Necessary and sufficient conditions of the existence of bounded solutions are obtained under assumption that the homogeneous equation admits an exponential dichotomy on the semi-axes.

Keywords: Moore-Penrose pseudoinvertible operator, exponential dichotomy, quantuum chaos.

1 Introduction

Boundary-value problems for the Schrödinger equation in a Hilbert space are encountered in quantum mechanics, quantuum functional analysis and other fields of physics and mathematics. The paper deals with the problem of existence and construction of the solutions of boundary-value problems for a nonstationary Schrödinger equation in a Hilbert space. The investigation of the nonstationary equation is important because, in numerous cases, it is necessary to find the applications in quantum functional analysis and differential equations [1].

2 Statement of the problem

In a Hilbert space $\mathcal{H}$, we consider the following boundary value problem

$$\frac{d\varphi(t)}{dt} = -iH(t)\varphi(t) + f(t), t \in \mathbb{R} \tag{1}$$

$$l\varphi(\cdot) = \alpha. \tag{2}$$

Unbounded operator $H(t)$ has the form $H(t) = H_0 + V(t)$. Here, $H_0 = H_0^*$ is a selfadjoint operator with dense domain of definition $D = D(H_0) \subset \mathcal{H}$ and the mapping $t \to V(t)$ is strongly continuous, l is linear and bounded operator which translates bounded solutions of (1) into the Hilbert space $\mathcal{H}_1$. We define an operator-valued function

$$\widetilde{V}(t) = e^{itH_0} V(t) e^{-itH_0}.$$

In this case, the Dyson representation is true for $\widetilde{V}(t)$ and we can find the evolutionary operator $\widetilde{U}(t,s)$. If $U(t,s) = e^{-itH_0}\widetilde{U}(t,s)e^{isH_0}$ then $\psi_s(t) = U(t,s)\psi$ is a weak solution of the homogeneous equation

$$\frac{d\varphi_0}{dt} = -iH(t)\varphi_0(t) \tag{3}$$

with condition $\psi_s(s) = \psi$ in a sense that, for any $\eta \in D$, the function $(\eta, \psi_s(t))$ is differentiable and

$$\frac{d}{dt}(\eta, \psi_s(t)) = -i(H_0\eta, \psi_s(t)) - i(V(t)\eta, \psi_s(t)), t, s \in \mathbb{R}.$$

Definition 1 [2]. *Evolution operator $\{U(t,s)|\ t \geq s;\ t,s \in J\}$ admits an exponential dichotomy on J, if there exist projector-valued function $\{P(t)|t \in J\}$ in $\mathcal{L}(\mathbf{B})$ and real constants $\alpha > 0$ and $M \geq 1$ such that:*

$$(i)\quad U(t,s)P(s) = P(t)U(t,s), \quad t \geq s;$$

(ii) Restriction $U(t,s) \restriction_{N(P(s))}, t \geq s$ of operator $U(t,s)$ onto kernel of $N(P(s))$ of projector $P(s)$ is an isomorphism from $N(P(s))$ onto $N(P(t))$. Define $U(s,t)$, as inverse

$$U(s,t) = (U(t,s) \restriction_{N(P(s))})^{-1} : N(P(s)) \to N(P(s))$$

$$(iii)\quad \|U(t,s)P(s)\| \leq Me^{-\alpha(t-s)}, \quad t \geq s;$$

$$(iiii)\quad \|U(t,s)(I - P(s))\| \leq Me^{-\alpha(s-t)}, \quad s \geq t.$$

3 Bounded solutions

Lemma. *Let $\{U(t,s),\ t \geq s \in \mathbb{R}\}$ is strongly continuous evolution operator of equation (3). Suppose that the following conditions are true:*

1. Operator $U(t,s)$ admits an exponential dichotomy on the semi-axes $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ with projector-valued functions $P_+(t)$ and $P_-(t)$, respectively.

2. Operator $D = P_+(0) - (I - P_-(0))$ has Moore-Penrose pseudoinvertible [3].

Then the following assertions are true.

1. There exist weak solutions of equation (1), bounded on the whole axis if and only if the vector-function $f \in BC(\mathbb{R}, \mathcal{H})$ satisifies condition

$$\int_{-\infty}^{+\infty} H(t)f(t)dt = 0, \tag{4}$$

where $H(t) = \mathcal{P}_{N(D^*)}P_-(0)U(0,t)$.

2. Under condition (4), weak solutions of the equation (1), bounded on the whole axis, has the form

$$\varphi_0(t,c) = U(t,0)P_+(0)\mathcal{P}_{N(D)}c + (G[f])(t,0) \quad \forall c \in \mathcal{H}, \tag{5}$$

where

$$(G[f])(t,s) = \begin{cases} \int_s^t U(t,\tau)P_+(\tau)f(\tau)d\tau - \int_t^{+\infty} U(t,\tau)(I-P_+(\tau))f(\tau)d\tau + \\ \quad +U(t,s)P_+(s)D^+[\int_s^{\infty} U(s,\tau)(I-P_+(\tau))f(\tau)d\tau + \\ \quad + \int_{-\infty}^{s} U(s,\tau)P_-(\tau)f(\tau)d\tau], \quad t \geq s \\ \int_{-\infty}^t U(t,\tau)P_-(\tau)f(\tau)d\tau - \int_t^{s} U(t,\tau)(I-P_-(\tau))f(\tau)d\tau + \\ \quad +U(t,s)(I-P_-(s))D^+[\int_s^{\infty} U(s,\tau)(I-P_+(\tau))f(\tau)d\tau + \\ \quad + \int_{-\infty}^{s} U(s,\tau)P_-(\tau)f(\tau)d\tau], \quad s \geq t \end{cases}$$

Theorem. *Let $Q = lU(\cdot,s)$. Boundary value problem (1), (2) has:*

1) bounded strong generalized solutions if and only if the following condition is true

$$\mathcal{P}_{N(\overline{Q}^*)}(\alpha - l(G[f])(\cdot,0)) = 0; \tag{6}$$

if $\alpha - l(G[f])(\cdot, 0) \in R(Q)$, *then solutions will be classical bounded;*

a2) bounded pseudosolutions if and only if the following condition is true

$$\mathcal{P}_{N(\overline{Q}^*)}(\alpha - l(G[f])(\cdot, 0)) \neq 0; \tag{7}$$

b) under condition of solvability (6) or (7) bounded solutions has the following form

$$\varphi_0(t, c) = U(t, 0)P_+(0)\mathcal{P}_{N(D)}\mathcal{P}_{N(Q)}\overline{c} + \overline{(G_1[f, \alpha])}(t, 0), \quad \forall \overline{c} \in \overline{\mathcal{H}},$$

where generalized Green operator $\overline{(G_1[f, \alpha])}(t, 0)$ *has the form*

$$\overline{(G_1[f, \alpha])}(t, s) = U(t, s)P_+(s)\mathcal{P}_{N(D)}\overline{Q}^+(\alpha - l(G[f])(\cdot, s)) + (G[f])(t, s).$$

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Computation of common Hilbert series for the differential system $s(1, 3, 5, 7)$ using the residue theorem

Victor Pricop

Abstract

Till now the Hilbert series was computing using the generalized Sylvester method that is not always simple. Getting a new, simpler methods for obtaining these series is welcome. This work is about on calculation of common Hilbert series for the differential system $s(1, 3, 5, 7)$ using the residue theorem.

Keywords: Hilbert series, Sibirsky algebra, Krull dimension).

1 Introduction

An important tool in the qualitative study of two-dimensional systems of differential equations is the Hilbert series of Sibirsky algebras.

One problem with Hilbert series corresponding to the differential systems is to determine a relations between them. In the work [1] were found some relations between generalized Hilbert series of differential systems with homogeneous nonlinearities of odd degree.

Before finding some relations between common Hilbert series of differential systems with homogeneous nonlinearities of odd degree they should be built. The construction of Hilbert series with generalized Sylvester method [2] is not always simple. From [3] it is known a method of computing common Hilbert series of invariants ring using the residues. This method was adapted for computation of common Hilbert series for Sibirsky algebras of comitants and invariants of differential systems. By this method, some unknown till now common Hilbert series of algebras of comitants and invariants was obtained. Computation more Hilbert series will help us find easier relationships between these series.

A common Hilbert series of the differential systems $s(1)$, $s(3)$, $s(5)$, $s(7)$, $s(1,3)$, $s(1,5)$, $s(1,7)$, $s(1,3,5)$, $s(1,3,7)$ are known. So, a computation of other series of the combination of these systems present an special interest.

2 Computation of common Hilbert series for the differential system $s(1,3,5,7)$

Let G be a linearly reductive group over an algebraically closed field K and V an n-dimensional rational representation. Through $H(K[V]^G,t)$ is denoted the Hilbert series of invariants ring $K[V]^G$ [5].

From [5] is known

Theorem 1.

$$H(K[V]^G,t)=\frac{1}{2\pi i}\int\limits_{S^1}\frac{1}{det(I-t_{\rho_V}(z))}\frac{dz}{z}, \tag{1}$$

where $S^1\subset\mathbb{C}$ is the unit circle $\{z:|z|=1\}$.

This formula is used on computing the common Hilbert series of invariants rings.

Using the Residue Theorem and corresponding generating function [4] the formula (1) can be adapted for computing the common Hilbert series for the Sibirsky algebras of comitants and invariants of differential systems as follows

Theorem 2.

$$H_{SI_\Gamma}(t)=\frac{1}{2\pi i}\int\limits_{S^1}\frac{\varphi_\Gamma^{(0)}(z)}{z}dz,$$

where $S^1 \subset \mathbb{C}$ *is the unit circle* $\{z : |z| = 1\}$, $\varphi_\Gamma^{(0)}(z)$ *is the corresponding generating function* [4], $\varphi_\Gamma^{(0)}(z) = (1-z^{-2})\psi_{m_0}^{(0)}(z)\psi_{m_1}^{(0)}(z)...\psi_{m_\ell}^{(0)}(z)$,

$$\psi_{m_i}^{(0)}(z) = \begin{cases} \frac{1}{(1-zt)(1-z^{-1}t)} & \text{for } m_i = 0, \\ \frac{1}{(1-z^{m_i+1}t)(1-z^{-m_i-1}t)\prod_{k=1}^{m_i}(1-z^{m_i-2k+1}t)^2} & \text{for } m_i \neq 0, \end{cases}$$

$\Gamma = \{m_i\}_{i=0}^{\ell}$ *and consists of a finite number* ($\ell < \infty$) *of distinct natural numbers.*

In contrast to the construction methods of Hilbert series, exposed in [4], using the theorem 2 it was obtained a common Hilbert series for the Sibirsky graded algebras of comitants $S_{1,3,5,7}$ and invariants $SI_{1,3,5,7}$ of the differential system $s(1,3,5,7)$.

Theorem 3. *For the differential system* $s(1,3,5,7)$ *the following common Hilbert series for Sibirsky algebras of comitants* $S_{1,3,5,7}$ *and invariants* $SI_{1,3,5,7}$ *was obtained*

$$H_{S_{1,3,5,7}}(t) = \frac{U(t) + 2298270315143980746t^{60} + t^{120}\,U(t^{-1})}{(1-t)^{14}(1+t)^{19}(1+t^2)^8(1-t^3)^{12}(1-t^5)^8(1-t^7)^4(1-t^9)},$$

where

$$U(t) = 1 + 6t + 20t^2 + 87t^3 + 642t^4 + 4481t^5 + 26793t^6 + 141973t^7 +$$
$$+684115t^8 + 3033350t^9 + 12465139t^{10} + 47749507t^{11} + 171414077t^{12} +$$
$$+579433144t^{13} + 1852114710t^{14} + 5618767624t^{15} + 16230539293t^{16} +$$
$$+44770726947t^{17} + 118233818156t^{18} + 299625404135t^{19} +$$
$$+730145608913t^{20} + 1714167261299t^{21} + 3883773551652t^{22} +$$
$$+8505306230645t^{23} + 18029418149708t^{24} + 37042309655531t^{25} +$$
$$+73851959357894t^{26} + 143039363140182t^{27} + 269416219454043t^{28}$$
$$+493944596168225t^{29} + 882268074320900t^{30} +$$
$$+1536543007952396t^{31} + 2611196867637156t^{32} +$$
$$+4333024660344442t^{33} + 7025611335473678t^{34} +$$

$$+11137398421309529t^{35} + 17271787147116907t^{36} +$$
$$+26216525599773850t^{37} + 38968364210329669t^{38} +$$
$$+56747752371861786t^{39} + 80997424826732157t^{40} +$$
$$+113358368681589288t^{41} + 155617153462411693t^{42} +$$
$$+209620178940739772t^{43} + 277153165150321324t^{44} +$$
$$+359788117447054402t^{45} + 458704770582751394t^{46} +$$
$$+574498645384155800t^{47} + 706992640391687667t^{48} +$$
$$+855072713288320920t^{49} + 1016569872742669961t^{50} +$$
$$+1188209740459545784t^{51} + 1365646993055807450t^{52} +$$
$$+1543595104982837472t^{53} + 1716052512321252802t^{54} +$$
$$+1876615582976246945t^{55} + 2018857942986265569t^{56} +$$
$$+2136746272693569424t^{57} + 2225056091622875140t^{58} +$$
$$+2279748435060291614t^{59},$$

$$H_{SI_{1,3,5,7}}(t) = \frac{V(t) + 32933502505147932t^{52} + t^{104}V(t^{-1})}{(1-t)^{15}(1+t)^{19}(1+t^2)^9(1-t^3)^{12}(1-t^5)^7(1-t^7)^3},$$

where

$$V(t) = 1 + 5t + 15t^2 + 70t^3 + 546t^4 + 3691t^5 + 21211t^6 + 108097t^7 +$$
$$+501215t^8 + 2135708t^9 + 8420376t^{10} + 30894213t^{11} + 106057925t^{12} +$$
$$+342316946t^{13} + 1043225615t^{14} + 3012988906t^{15} + 8273667765t^{16} +$$
$$+21663519624t^{17} + 54225659702t^{18} + 130054129145t^{19} +$$
$$+299492368986t^{20} + 663439513913t^{21} + 1416140486098t^{22} +$$
$$+2917219852903t^{23} + 5807630254373t^{24} + 11187994444298t^{25} +$$
$$+20880385856690t^{26} + 37794195363608t^{27} + 66411190209119t^{28} +$$
$$+113391841520052t^{29} + 188282608991333t^{30} +$$
$$+304271520124478t^{31} + 478898737877115t^{32} +$$
$$+734584562409596t^{33} + 1098797608776741t^{34} +$$

$$+1603661779481979t^{35} + 2284804664001899t^{36}+$$
$$+3179293473234493t^{37} + 4322594520474429t^{38}+$$
$$+5744627532607767t^{39} + 7465155325802975t^{40}+$$
$$+9488929831214829t^{41} + 11801175204390804t^{42}+$$
$$+14364091127469868t^{43} + 17115070624832596t^{44}+$$
$$+19967223601230372t^{45} + 22812575427180540t^{46}+$$
$$+25527987499683011t^{47} + 27983465544664079t^{48}+$$
$$+30052140716959960t^{49} + 31620895669339212t^{50}+$$
$$+32600424240909358t^{51}.$$

From this theorem result that the Krull dimension [4] of Sibirsky graded algebra $S_{1,3,5,7}$ (respectively $SI_{1,3,5,7}$) is equal to 39 (respectively 37).

We note that the Krull dimension plays an important role in solving the center-focus problem for the differential system $s(1,3,5,7)$ [6].

Remark. *We note that the Hilbert series of Sibirsky graded algebra of comitants* $H_{S_\Gamma}(t) = H_{SI_{\Gamma\cup\{0\}}}(t),$ *where* $\Gamma = \{m_1, m_2, ..., m_\ell\} \not\ni \{0\}$.

3 Conclusion

In this paper was computing the common Hilbert series of Sibirsky graded algebras for the differential system $s(1,3,5,7)$. This method is easier than generalized Sylvester method, but only allows for the computation of common Hilbert series of Sibirsky graded algebras of invariants and comitants of differential systems, but it also allows us to obtain some new unknown series till now.

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Quartic differential systems with an affine real invariant straight line of algebraic multiplicity two and three

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Abstract

In this paper we give the coefficient conditions when the differential polynomial system of the fourth degree has an affine real invariant straight line of algebraic multiplicity two (three).

Keywords: quartic differential system, invariant straight line, algebraic multiplicity.

1 Introduction

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

and the vector field $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to system (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$.

A curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ is said to be an *invariant algebraic curve* of (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $deg(K_f) \leq n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y)$ holds.

Definition 1 [1] *An invariant algebraic curve f of degree d for the vector field $\mathbb{X}$ has algebraic multiplicity m when m is the greatest* *positive integer such that the m-th power of f divides $E_d(\mathbb{X})$, where*

$$E_d(\mathbb{X}) = det \begin{pmatrix} \upsilon_1 & \upsilon_2 & ... & \upsilon_k \\ \mathbb{X}(\upsilon_1) & \mathbb{X}(\upsilon_2) & ... & \mathbb{X}(\upsilon_k) \\ ... & ... & ... & ... \\ \mathbb{X}^{k-1}(\upsilon_1) & \mathbb{X}^{k-1}(\upsilon_2) & ... & \mathbb{X}^{k-1}(\upsilon_k) \end{pmatrix},$$

and $\upsilon_1, \upsilon_2, ..., \upsilon_k$ is a basis of $\mathbb{C}_d[x, y]$.

If $d = 1$ then $\upsilon_1 = 1$, $\upsilon_2 = x$, $\upsilon_3 = y$ and

$$E_1(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P).$$

2 The coefficient conditions when the differential system of the fourth degree has an affine real invariant straight line of algebraic multiplicity two and three

We consider the differential system of the fourth degree

$$\begin{aligned} \dot{x} &= P_0 + P_1(x, y) + P_2(x, y) + P_3(x, y) + P_4(x, y) \equiv P(x, y), \\ \dot{y} &= Q_0 + Q_1(x, y) + Q_2(x, y) + Q_3(x, y) + Q_4(x, y) \equiv Q(x, y), \end{aligned} \tag{2}$$

where P_k and $Q_k,\ k = 1, 2, 3, 4$ are homogeneous polynomials in x and y of degree k.

Suppose that

$$yP_4(x, y) - xQ_4(x, y) \not\equiv 0, \qquad \gcd(P, Q) = 1, \tag{3}$$

i.e. at infinity the system (2) has at most five distinct singular points and the right-hand sides of (2) do not have the common divisors of degree greatest that 0.

Let the system (2) has a real invariant straight line l. By an affine transformation we can make l to be described by the equation $x = 0$.

Then, the system (2) looks as:

$$\begin{aligned}
&\dot{x} = x(a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3+ \\
&\quad +a_8x^2y + a_9xy^2 + a_{10}y^3), \\
&\dot{y} = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y+ \\
&+b_8xy^2 + b_9y^3 + b_{10}x^4 + b_{11}x^3y + b_{12}x^2y^2 + b_{13}xy^3 + b_{14}y^4.
\end{aligned} \tag{4}$$

For (4) the determinant $E_1(\mathbb{X})$ is a polynomial in x and y of degree 11.

Theorem 1 *For quartic differential system* $\{(4), (3)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is greater or equal two if and only if at least one of the following five sets of conditions:*

1) $a_1 = a_3 = a_6 = a_{10} = 0$;

2) $a_3 = a_6 = a_{10} = b_2 - a_1 = b_5 = b_9 = b_{14} = 0,\ a_1 \neq 0$;

3) $a_6 = a_{10} = 0$, $b_0 = (-a_1^2 + a_1b_2)/a_3$, $b_5 - a_3 = b_9 = b_{14} = 0$;

4) $a_{10} = 0$, $b_0 = (a_1(b_5 - a_3))/a_6$, $b_2 = (-a_3^2 + a_1a_6 + a_3b_5)/a_6$, $b_9 - a_6 = b_{14} = 0$;

5) $b_0 = (a_1(b_9 - a_6))/a_{10}$, $b_2 = (a_1a_{10} - a_3a_6 + a_3b_9)/a_{10}$, $b_5 = (a_{10}a_3 - a_6^2 + a_6b_9)/a_{10}$, $b_{14} = a_{10}$

is satisfied.

Theorem 2 *For quartic differential system* $\{(4), (3)\}$ *the algebraic multiplicity of the invariant straight line* $x = 0$ *is greater or equal three if and only if at least one of the following twelve sets of conditions:*

1) $a_1 = a_2 = a_3 = a_5 = a_6 = a_9 = a_{10} = 0$;

2) $a_1 = a_3 = 0$, $a_5 = a_2b_2/b_0$, $a_6 = 0$, $a_9 = a_2b_5/b_0$, $a_{10} = b_9 = b_{14} = 0$, $a_2 \neq 0$;

3) $a_1 = a_3 = a_6 = 0$, $a_9 = a_5b_5/b_2$, $a_{10} = 0, b_0 = a_2b_2/a_5$, $b_9 = b_{14} = 0$;

4) $a_1 = a_2 = a_3 = a_5 = a_6 = a_{10} = b_0 = b_2 = b_9 = b_{14} = 0$, $a_9 \neq 0$;

5) $a_1 = a_3 = a_5 = a_6 = a_9 = a_{10} = b_2 = b_5 = b_9 = b_{14} = 0$;

6) $a_2 = (-a_5b_0 + a_1b_4)/a_1$, $a_3 = a_6 = a_{10} = 0$, $b_2 - a_1 = b_5 = 0$, $b_1 = (b_0(-a_5b_0 + a_1b_4))/a_1^2$, $b_8 = (a_1a_5 + a_9b_0)/a_1$, $b_9 = 0$, $b_{13} - a_9 = b_{14} = 0$;

7) $a_6 = a_{10} = 0$, $b_0 = (a_1(b_2 - a_1))/a_3$, $b_2 = 2a_1$, $b_5 = a_3$, $b_8 = (a_3a_5 + a_1a_9)/a_3$, $b_4 = (a_2a_3 + a_1a_5)/a_3$, $b_9 = 0$, $b_{13} - a_9 = b_{14} = 0$;

8) $a_6 = a_{10} = 0$, $b_0 = (a_1(b_2 - a_1))/a_3$, $b_1 = (a_2(b_2 - a_1))/a_3$, $b_4 = (a_2a_3 - a_1a_5 + a_5b_2)/a_3$, $b_8 = (a_3a_5 - a_1a_9 + a_9b_2)/a_3$, $b_5 = a_3$, $b_9 =$

$b_{13} - a_9 = b_{14} = 0$, $b_2 - 2a_1 \neq 0$;

9) $a_1 = ((b_5 - a_3)(2a_3 - b_5))/a_6$, $b_0 = ((a_3 - b_5)^2(2a_3 - b_5))/a_6^2$, $b_1 = (2a_3^2a_5 - 3a_2a_3a_6 + 2a_3a_6b_4 - 3a_3a_5b_5 + 2a_2a_6b_5 - a_6b_4b_5 + a_5b_5^2)/a_6^2$, $b_2 = ((b_5 - a_3)(3a_3 - b_5))/a_6$, $b_8 = (a_5a_6 - a_3a_9 + a_9b_5)/a_6$, $b_9 - a_6 = b_{13} - a_9 = b_{14} = 0$, $a_{10} = 0$;

10) $a_{10} = 0$, $b_0 = (a_1(b_5 - a_3))/a_6$, $b_1 = (a_2(b_5 - a_3))/a_6$, $b_2 = (-a_3^2 + a_1a_6 + a_3b_5)/a_6$, $b_4 = (-a_3a_5 + a_2a_6 + a_5b_5)/a_6$, $b_8 = (a_5a_6 - a_3a_9 + a_9b_5)/a_6, b_9 - a_6 = b_{13} - a_9 = b_{14} = 0$;

11) $b_0 = (a_1(b_9 - a_6))/a_{10}$, $b_2 = (a_1a_{10} - a_3a_6 + a_3b_9)/a_{10}$, $b_1 = (a_2(b_9 - a_6))/a_{10}$, $b_4 = (a_{10}a_2 - a_5a_6 + a_5b_9)/a_{10}$, $b_5 = (a_{10}a_3 - a_6^2 + a_6b_9)/a_{10}$, $b_8 = (a_{10}a_5 - a_6a_9 + a_9b_9)/a_{10}$, $b_{13} = a_9, b_{14} = a_{10}$;

12) $a_1 = ((b_9 - a_6)(a_{10}a_3 + 2a_6^2 - 3a_6b_9 + b_9^2))/a_{10}^2$, $b_0 = (-a_1a_6 + a_1b_9)/a_{10}$, $b_2 = (a_1a_{10} - a_3a_6 + a_3b_9)/a_{10}$, $b_1 = -(a_{10}^2a_3a_5 + a_{10}^2a_2a_6 + 2a_{10}a_5a_6^2 - a_{10}a_3a_6a_9 - 2a_6^3a_9 - a_{10}^2a_3b_8 - 2a_{10}a_6^2b_8 - a_{10}^2a_2b_9 - 3a_{10}a_5a_6b_9 + a_{10}a_3a_9b_9 + 5a_6^2a_9b_9 + 3a_{10}a_6b_8b_9 + a_{10}a_5b_9^2 - 4a_6a_9b_9^2 - a_{10}b_8b_9^2 + a_9b_9^3)/a_{10}^3$, $b_4 = (a_{10}^2a_2 - 3a_{10}a_5a_6 + 2a_6^2a_9 + 2a_{10}a_6b_8 + 2a_{10}a_5b_9 - 3a_6a_9b_9 - a_{10}b_8b_9 + a_9b_9^2)/a_{10}^2$, $b_5 = (a_{10}a_3 - a_6^2 + a_6b_9)/a_{10}$, $b_{13} = a_9$, $b_{14} = a_{10}$

hold.

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Improved Direct and Inverse Theorems in Weighted Lebesgue Spaces with Variable Exponent

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Abstract

The improved direct, inverse and simultaneous theorems of approximation theory in the weighted variable exponent Lebesgue spaces in the term of the fractional order modulus of smoothness are obtained.

Keywords: Direct theorems, inverse theorems, Muckenhoupt weights, fractional modulus of smoothness, simultaneous approximation.

1 Introduction

Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [0, \infty)$ be a Lebesgue measurable 2π periodic function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is defined as the set of all Lebesgue measurable 2π periodic functions f such that $\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx < \infty$. During this work we suppose that the considered exponent functions $p(\cdot)$ satisfy the conditions

$$1 \le p_- := ess\inf_{x\in\mathbb{T}}\ p(x) \le ess\sup_{x\in\mathbb{T}}\ p(x) := p^+ < \infty,$$
$$|p(x) - p(y)| \ln(1/|x-y|) \le c(p) < \infty, x, y \in \mathbb{T}, 0 < |x-y| \le 1/2.$$

The class of these exponents we denote by $\mathcal{P}(\mathbb{T})$. If $p(\cdot) \in \mathcal{P}(\mathbb{T})$ and in addition $p_- > 1$, then we say that $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. Equipped with the norm $\|f\|_{p(\cdot)} = \left\{\inf \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\right\}$ the space $L^{p(\cdot)}(\mathbb{T})$ becomes a Banach space.

Let ω be a weight function on $\mathbb{T}$, i.e. an almost everywhere positive and Lebesgue integrable function on $\mathbb{T}$. For a given weight ω we define the weighted variable exponent Lebesgue space $L_{\omega}^{p(\cdot)}(\mathbb{T})$ as the set of all measurable functions f on $\mathbb{T}$ such that $f\omega \in L^{p(\cdot)}(\mathbb{T})$. The norm of $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ can be defined as $\|f\|_{p(\cdot),\omega} := \|f\omega\|_{p(\cdot)}$. In our discussions we assume that $\omega \in A_{p(\cdot)}(\mathbb{T})$.

Definition 1 *We say that $\omega \in A_{p(\cdot)}(\mathbb{T})$ if the inequality*

$$\sup_{I\subset\mathbb{T}} |I|^{-1} \|\omega\chi_I\|_{p(\cdot)} \left\|\omega^{-1}\chi_I\right\|_{p'(\cdot)} < \infty,\ 1/p(\cdot) + 1/p'(\cdot) = 1,$$

holds, where $|I|$ is the Lebesgue measure of the interval $I \subset \mathbb{T}$ with the characteristic function χ_I.

Let $f \in L^1(\mathbb{T})$ with $\int_0^{2\pi} f(x)\,dx = 0$. For $\alpha \in \mathbb{R}^+$ the αth integral of f is defined by $I_\alpha(f,x) := \sum\limits_{k\in\mathbb{Z}^*} c_k(f)(ik)^{-\alpha} e^{ikx}$, where $(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i\alpha\ sign\ k}$, $\mathbb{Z}^* := \{\pm1, \pm2, \pm3, ...\}$ and c_k, $k \in \mathbb{Z}^*$, are the Fourier coefficients of f with respect to exponential system. For $\alpha \in (0,1)$ let $f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(f,x)$. If $r \in \mathbb{R}^+$ with integer part $[r]$, and $\alpha := r - [r]$, then the rth derivative of f is defined by $f^{(r)}(x) := \left(f^{(\alpha)}(x)\right)^{([r])} = \frac{d^{[r]+1}}{dx^{[r]+1}} I_{1-\alpha}(f,x)$ if the right sides exist [1, p. 347]. Let $x,t \in \mathbb{R}$, $r \in \mathbb{R}^+$ and let $\Delta_t^r f(x) := \sum_{k=0}^{\infty} (-1)^k [C_k^r] f(x + (r-k)t)$ for $f \in L^1(\mathbb{T})$, where $[C_k^r] := r(r-1)(r-2)...(r-k+1)/k!$ for $k > 1$, $[C_k^r] := r$ for $k = 1$ and $[C_k^r] := 1$ for $k = 0$.

Definition 2 *Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and $r \in \mathbb{R}^+$. We define the rth modulus of smoothness as*

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \sup_{|h|\le\delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r f(x)\,dt \right\|_{p(\cdot),\omega}, \quad \delta > 0.$$

Clearly $\Omega(f,\delta)_{p(\cdot),\omega}$ is well defined because by Theorem on the boundedness of maximal function in $L_{\omega}^{p(\cdot)}(\mathbb{T})$ proved in [2] we have $\Omega_r(f,\delta)_{p(\cdot),\omega} \le c(p)\|f\|_{p(\cdot),\omega}$.

2 Main Results

Let $S_n(f)$ be the nth partial sum of the Fourier series of $f \in L_\omega^{p(\cdot)}(\mathbb{T})$ and let $W_{\omega,\beta}^{p(\cdot)}(\mathbb{T}) := \left\{ f \in L_\omega^{p(\cdot)}(\mathbb{T}) : \ f^{(\beta)} \in L_\omega^{p(\cdot)}(\mathbb{T}) \text{ for } \beta \in \mathbb{R}^+ \right\}$ be the weighted variable exponent Sobolev space. By $c(\cdot)$, $c(\cdot,\cdot)$, $c(\cdot,\cdot,\cdot)$ we denote the constants depending in general on the parameters given in the brackets. Our main results are following.

Theorem 1 *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and let $r \in \mathbb{R}^+$. If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, then there exists a constant $c(p,r)$ such that*

$$\frac{c(p,r)}{n^r}\left\{\sum_{\nu=1}^{n} \nu^{\beta r-1} E_\nu^\beta(f)_{p(\cdot),\omega}\right\}^{1/\beta} \leq \Omega_r(f,1/n)_{p(\cdot),\omega}\ , \ n=1,2,...$$

where $\beta := \max\{2, p_+\}$.

Theorem 2 *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and let $r \in \mathbb{R}^+$. If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, then there exists a constant $c(p,r)$ such that*

$$\Omega_r(f,1/n)_{p(\cdot),\omega} \leq \frac{c(p,r)}{n^r}\left\{\sum_{\nu=0}^{n} (\nu+1)^{\gamma r-1} E_\nu^\gamma(f)_{p(\cdot),\omega}\right\}^{1/\gamma}, n=1,2,...$$

where $\gamma := \min\{2, p_-\}$.

Theorem 3 *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$. If $\sum\limits_{k=1}^{\infty} k^{\gamma r-1} E_\nu^\gamma(f)_{p(\cdot),\omega} < \infty$ for some $r \in \mathbb{R}^+$and $\gamma := \min\{2, p_-\}$, then $f \in W_{\omega,r}^{p(\cdot)}(\mathbb{T})$ and there exists a constant $c(p,r)$ such that for every $n = 1,2,3,...$,*

$$E_n\left(f^{(r)}\right)_{p(\cdot),\omega} \leq c(p,r)\left(n^r E_n(f)_{p(\cdot),\omega} + \left\{\sum_{\nu=n+1}^{\infty} \nu^{\gamma r-1} E_\nu^\gamma(f)_{p(\cdot),\omega}\right\}^{1/\gamma}\right).$$

Corollary 1 *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}(\mathbb{T})$ and let $r \in \mathbb{R}^+$. If $f \in W_{\omega,r}^{p(\cdot)}(\mathbb{T})$ and $\beta := \max\{2, p_+\}$, then there exists a constant*

$c(p,r)$ such that for every $n=1,2,3,...$

$$\left\{\sum_{\nu=1}^{n}\nu^{\beta r-1}E_{\nu}^{\beta}\left(f\right)_{p(\cdot),\omega}\right\}^{1/\beta}\leq c\left(p,r\right)\left\|f^{(r)}\right\|_{p(\cdot),\omega}.$$

Corollary 2 *Let $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}(\mathbb{T})$ and let $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$. If $\sum\limits_{k=1}^{\infty}k^{\gamma\alpha-1}E_{\nu}^{\gamma}(f)_{p(\cdot),\omega}$ for some $\alpha\in\mathbb{R}^{+}$and $\gamma:=\min\{2,p_{-}\}$, then there exists a positive constant $c(p,\alpha,r)$ such that for every $n=1,2,3,...$ and $r\in\mathbb{R}^{+}$*

$$\Omega_r\left(f^{(\alpha)},1/n\right)_{p(\cdot),\omega}\leq c\left(p,\alpha,r\right)\left(\left\{\sum_{\nu=n+1}^{\infty}\nu^{\gamma\alpha-1}E_{\nu}^{\gamma}\left(f\right)_{p(\cdot),\omega}\right\}^{1/\gamma}+\right.$$

$$\left.+\frac{1}{n^r}\left\{\sum_{\nu=1}^{n}\nu^{\gamma(r+\alpha)-1}E_{\nu}^{\gamma}\left(f\right)_{p(\cdot),\omega}\right\}^{1/\gamma}\right).$$

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Morphisms and antimorphisms of Boolean evolution and antievolution functions

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Abstract

The Boolean evolution and antievolution functions model the asynchronous circuits from electronics. Our purpose is to introduce their morphisms and antimorphisms.

Keywords: Boolean function, morphism, antimorphism, evolution function, antievolution function

We denote $\mathbf{B}=\{0,1\}$ the binary Boole algebra and $\mathbf{N}_{\_}=\{-1,0,1,...\}$. Let $\Phi,\Psi,h,h':\mathbf{B}^n\rightarrow\mathbf{B}^n$, for which we define $\forall i\in\{1,...,n\},\forall\nu\in\mathbf{B}^n,\forall\mu\in\mathbf{B}^n,\Phi_i^{\nu}(\mu)=\begin{cases}\mu_i, if\ \nu_i=0,\\ \Phi_i(\mu), if\ \nu_i=1\end{cases}$. If $\forall\nu\in\mathbf{B}^n,\forall\mu\in\mathbf{B}^n$, $h(\Phi^{\nu}(\mu))=\Psi^{h'(\nu)}(h(\mu))$, we say that the **morphism** (h,h') is defined, from Φ to Ψ and if $\forall\nu\in\mathbf{B}^n,\forall\mu\in\mathbf{B}^n$, $h(\mu)=\Psi^{h'(\nu)}(h(\Phi^{\nu}(\mu)))$, we say that the **antimorphism** $(h,h')^{\sim}$ is defined, from Φ to Ψ. The sets of the morphisms and of the antimorphisms from Φ to Ψ are denoted with $Hom(\Phi,\Psi),Hom^{\sim}(\Phi,\Psi)$. We denote $\widehat{\Pi}_n=\{\alpha|\alpha:\mathbf{N}\longrightarrow\mathbf{B}^n,\forall i\in\{1,...,n\},\{k|k\in\mathbf{N},\alpha_i^k:=\alpha_i(k)=1\}$ is infinite$\}$. The functions $\widehat{\Phi},\widehat{\Phi}^{\sim}$ given by $\mathbf{B}^n\times\mathbf{N}_{\_}\times\widehat{\Pi}_n\ni(\mu,k,\alpha)\longmapsto\widehat{\Phi}^{\alpha}(\mu,k),\widehat{\Phi}^{\sim\alpha}(\mu,k)\in\mathbf{B}^n$, $\widehat{\Phi}^{\alpha}(\mu,k)=\begin{cases}\mu,\ if\ k=-1,\\ \Phi^{\alpha^0}(\mu),\ if\ k=0,\\ (\Phi^{\alpha^k}\circ\Phi^{\alpha^{k-1}}\circ...\circ\Phi^{\alpha^0})(\mu),\ if\ k\geq 1\end{cases}$, $\widehat{\Phi}^{\sim\alpha}(\mu,k)=\begin{cases}\mu,\ if\ k=-1,\\ \Phi^{\alpha^0}(\mu),\ if\ k=0,\\ (\Phi^{\alpha^0}\circ\Phi^{\alpha^1}\circ...\circ\Phi^{\alpha^k})(\mu),\ if\ k\geq 1\end{cases}$ are called **evolution** and **anti-evolution function** and they model the asynchronous circuits, respectively the time reversed asynchronous circuits. We have by definition the **orbit** $\widehat{Or}^{\alpha}_{\Phi}(\mu) = \{\widehat{\Phi}^{\alpha}(\mu,k)|k \in \mathbf{N}_{\text{-}}\}$ and the **omega limit set** $\widehat{\omega}^{\alpha}_{\Phi}(\mu) = \{\lambda|\lambda \in \mathbf{B}^n, \{k|k \in \mathbf{N}_{\text{-}}, \widehat{\Phi}^{\alpha}(\mu,k) = \lambda\}$ is infinite$\}$ and similarly for $\widehat{\Phi}^{\sim}$. For $\alpha : \mathbf{N} \longrightarrow \mathbf{B}^n$ we also define $\widehat{h'}(\alpha) : \mathbf{N} \longrightarrow \mathbf{B}^n$ by $\forall k \in \mathbf{N}, \widehat{h'}(\alpha)^k = h'(\alpha^k)$ and $\Omega_n = \{h'|\widehat{h'}(\widehat{\Pi}_n) \subset \widehat{\Pi}_n\}$. Our purpose is to introduce the morphisms and the antimorphisms of evolution and antievolution functions.

Definition 1. We consider the functions $\Phi, \Psi, h, h' : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and we suppose that $h' \in \Omega_n$. We say that the couple (h,h') is a **morphism from** the evolution function $\widehat{\Phi}$ **to** the evolution function $\widehat{\Psi}$, denoted by $(h,h') : \widehat{\Phi} \longrightarrow \widehat{\Psi}$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_{\text{-}}, \forall \alpha \in \widehat{\Pi}_n$, $h(\widehat{\Phi}^{\alpha}(\mu,k)) = \widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu),k)$; (h,h') is a **morphism from** the antievolution function $\widehat{\Phi}^{\sim}$ **to** the antievolution function $\widehat{\Psi}^{\sim}$, denoted by $(h,h') : \widehat{\Phi}^{\sim} \longrightarrow \widehat{\Psi}^{\sim}$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_{\text{-}}, \forall \alpha \in \widehat{\Pi}_n$, $h(\widehat{\Phi}^{\sim\alpha}(\mu,k)) = \widehat{\Psi}^{\sim\widehat{h'}(\alpha)}(h(\mu),k)$. We denote with $Hom(\widehat{\Phi},\widehat{\Psi})$, $Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$ the previous sets of morphisms.

Theorem 1. We get $Hom(\widehat{\Phi},\widehat{\Psi}) = Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim}) = \{(h,h')|(h,h') \in Hom(\Phi,\Psi)$ and $h' \in \Omega_n\}$.

Theorem 2. For $\Gamma : \mathbf{B}^n \to \mathbf{B}^n$, we have $(h,h') \in Hom(\widehat{\Phi},\widehat{\Psi}), (g,g') \in Hom(\widehat{\Psi},\widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi},\widehat{\Gamma})$ and $(h,h') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$, $(g,g') \in Hom(\widehat{\Psi}^{\sim},\widehat{\Gamma}^{\sim}) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Gamma}^{\sim})$.

Definition 2. The morphism $(g \circ h, g' \circ h')$ is by definition the **composition** of (g,g') and (h,h') and its notation is $(g,g') \circ (h,h')$.

Theorem 3. Let $(h,h') \in Hom(\widehat{\Phi},\widehat{\Psi})$, $(g,g') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$. Then $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \widehat{\Pi}_n$, $h(\widehat{Or}^{\alpha}_{\Phi}(\mu)) = \widehat{Or}^{\widehat{h'}(\alpha)}_{\Psi}(h(\mu)), g(\widehat{Or}^{\sim\alpha}_{\Phi}(\mu)) = \widehat{Or}^{\sim\widehat{g'}(\alpha)}_{\Psi}(g(\mu)), h(\widehat{\omega}^{\alpha}_{\Phi}(\mu)) = \widehat{\omega}^{\widehat{h'}(\alpha)}_{\Psi}(h(\mu)), g(\widehat{\omega}^{\sim\alpha}_{\Phi}(\mu)) = \widehat{\omega}^{\sim\widehat{g'}(\alpha)}_{\Psi}(g(\mu))$.

Theorem 4. For any $\mu \in \mathbf{B}^n$ and any $\alpha \in \widehat{\Pi}_n$, if $\widehat{\Phi}^{\alpha}(\mu,\cdot)$ is periodic, with the period $p \geq 1 : \forall k \in \mathbf{N}_{\text{-}}, \widehat{\Phi}^{\alpha}(\mu,k) = \widehat{\Phi}^{\alpha}(\mu,k+p)$ and $(h,h') \in Hom(\widehat{\Phi},\widehat{\Psi})$, then $\widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu),\cdot)$ is periodic with the period p; if we suppose that $\widehat{\Phi}^{\sim\alpha}(\mu,\cdot)$ is periodic, with the period $p \geq 1 : \forall k \in \mathbf{N}_{\text{-}}, \widehat{\Phi}^{\sim\alpha}(\mu,k) = \widehat{\Phi}^{\sim\alpha}(\mu,k+p)$ and $(h,h') \in Hom(\widehat{\Phi}^{\sim},\widehat{\Psi}^{\sim})$, then $\widehat{\Psi}^{\sim\widehat{h'}(\alpha)}(h(\mu),\cdot)$ is periodic with the period p.

Theorem 5. Let $\mu \in \mathbf{B}^n$ and we suppose that $\Phi(\mu) = \mu$. If $(h, h') \in Hom(\widehat{\Phi}, \widehat{\Psi})$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall \alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_\_, \widehat{\Psi}^{\widehat{h'(\alpha)}}(h(\mu), k) = h(\mu)$; if $(h, h') \in Hom(\widehat{\Phi}^\sim, \widehat{\Psi}^\sim)$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall \alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_\_, \widehat{\Psi}^{\sim\widehat{h'(\alpha)}}(h(\mu), k) = h(\mu)$.

Definition 3. We ask that $h' \in \Omega_n$. We say that the couple (h, h') is an **antimorphism from** $\widehat{\Phi}^\sim$ **to** $\widehat{\Psi}$, denoted $(h, h')^\sim : \widehat{\Phi}^\sim \longrightarrow \widehat{\Psi}$ or simply $(h, h')^\sim$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_\_, \forall \alpha \in \widehat{\Pi}_n, h(\mu) = \widehat{\Psi}^{\widehat{h'(\alpha)}}(h(\widehat{\Phi}^{\sim\alpha}(\mu, k)), k)$ and (h, h') is by definition an **antimorphism from** $\widehat{\Phi}$ **to** $\widehat{\Psi}^\sim$, denoted $(h, h')^\sim : \widehat{\Phi} \longrightarrow \widehat{\Psi}^\sim$ or $(h, h')^\sim$, if $\forall \mu \in \mathbf{B}^n, \forall k \in \mathbf{N}_\_, \forall \alpha \in \widehat{\Pi}_n, h(\mu) = \widehat{\Psi}^{\sim\widehat{h'(\alpha)}}(h(\widehat{\Phi}^{\alpha}(\mu, k)), k)$. We use the notation $Hom^\sim(\widehat{\Phi}^\sim, \widehat{\Psi})$, $Hom^\sim(\widehat{\Phi}, \widehat{\Psi}^\sim)$ for the previous sets of antimorphisms.

Theorem 6. We get $Hom^\sim(\widehat{\Phi}^\sim, \widehat{\Psi}) = Hom^\sim(\widehat{\Phi}, \widehat{\Psi}^\sim) = \{(h, h')^\sim | (h, h')^\sim \in Hom^\sim(\Phi, \Psi) \text{ and } h' \in \Omega_n\}$.

Theorem 7. a) $(h, h')^\sim \in Hom^\sim(\widehat{\Phi}^\sim, \widehat{\Psi}), (g, g')^\sim \in Hom^\sim(\widehat{\Psi}, \widehat{\Gamma}^\sim) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi}^\sim, \widehat{\Gamma}^\sim)$, b) $(h, h')^\sim \in Hom^\sim(\widehat{\Phi}, \widehat{\Psi}^\sim), (g, g')^\sim \in Hom^\sim(\widehat{\Psi}^\sim, \widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h') \in Hom(\widehat{\Phi}, \widehat{\Gamma})$, c) $(h, h') \in Hom(\widehat{\Phi}, \widehat{\Psi})$, $(g, g')^\sim \in Hom^\sim(\widehat{\Psi}, \widehat{\Gamma}^\sim) \Longrightarrow (g \circ h, g' \circ h')^\sim \in Hom^\sim(\widehat{\Phi}, \widehat{\Gamma}^\sim)$, d) $(h, h') \in Hom(\widehat{\Phi}^\sim, \widehat{\Psi}^\sim), (g, g')^\sim \in Hom(\widehat{\Psi}^\sim, \widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h')^\sim \in Hom^\sim(\widehat{\Phi}^\sim, \widehat{\Gamma})$, e) $(h, h')^\sim \in Hom^\sim(\widehat{\Phi}^\sim, \widehat{\Psi}), (g, g') \in Hom(\widehat{\Psi}, \widehat{\Gamma}) \Longrightarrow (g \circ h, g' \circ h')^\sim \in Hom^\sim(\widehat{\Phi}^\sim, \widehat{\Gamma})$, f) $(h, h')^\sim \in Hom^\sim(\widehat{\Phi}, \widehat{\Psi}^\sim), (g, g') \in Hom(\widehat{\Psi}^\sim, \widehat{\Gamma}^\sim) \Longrightarrow (g \circ h, g' \circ h')^\sim \in Hom^\sim(\widehat{\Phi}, \widehat{\Gamma}^\sim)$ hold.

Definition 4. In a), b) the morphism $(g \circ h, g' \circ h')$ is by definition the **composition** of the antimorphisms $(g, g')^\sim$ and $(h, h')^\sim$ and its notation is $(g, g')^\sim \circ (h, h')^\sim$. In c), d) the antimorphism $(g \circ h, g' \circ h')^\sim$ is by definition the **composition** of the antimorphism $(g, g')^\sim$ with the morphism (h, h') and it has the notation $(g, g')^\sim \circ (h, h')$. Similarly for $(g \circ h, g' \circ h')^\sim$ denoted $(g, g') \circ (h, h')^\sim$ from e), f).

Theorem 8. Let the functions $\Phi, \Psi : \mathbf{B}^n \longrightarrow \mathbf{B}^n$ and the antimorphisms $(h, h')^\sim \in Hom(\widehat{\Phi}^\sim, \widehat{\Psi}), (g, g')^\sim \in Hom(\widehat{\Phi}, \widehat{\Psi}^\sim)$; $\forall \mu \in \mathbf{B}^n, \forall \alpha \in \widehat{\Pi}_n$, we have $\forall \nu \in \widehat{Or}_{\Phi}^{\sim\alpha}(\mu), h(\mu) \in \widehat{Or}_{\Psi}^{\widehat{h'(\alpha)}}(h(\nu)), \forall \nu \in \widehat{Or}_{\Phi}^{\alpha}(\mu), g(\mu) \in$

$\widehat{Or}_{\Psi}^{\sim\widehat{h'}(\alpha)}(g(\nu)), \forall\nu \in \widehat{\omega}_{\Phi}^{\sim\alpha}(\mu), h(\mu) \in \widehat{\omega}_{\Psi}^{\widehat{h'}(\alpha)}(h(\nu)), \forall\nu \in \widehat{\omega}_{\Phi}^{\alpha}(\mu), g(\mu) \in \widehat{\omega}_{\Psi}^{\sim\widehat{h'}(\alpha)}(g(\nu))$.

Theorem 9. Let $\mu \in \mathbf{B}^n$ with $\Phi(\mu) = \mu$. If $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi})$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall\alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \widehat{\Psi}^{\widehat{h'}(\alpha)}(h(\mu), k) = h(\mu)$ hold; if $(h, h')^{\sim} \in Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}^{\sim})$, then $\Psi(h(\mu)) = h(\mu)$ and $\forall\alpha \in \widehat{\Pi}_n, \forall k \in \mathbf{N}_{\_}, \widehat{\Psi}^{\sim\widehat{h'}(\alpha)}(h(\mu), k) = h(\mu)$ are true.

Remark 1. The next sets $Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}), Hom(\widehat{\Phi}, \widehat{\Psi}^{\sim})$, $Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi})$, $Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi}^{\sim})$ are defined like in Definition 1 and Definition 3. We can prove that $Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}) = Hom(\widehat{\Phi}, \widehat{\Psi}^{\sim}), Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}) \subset Hom(\Phi, \Psi), Hom(\widehat{\Phi}^{\sim}, \widehat{\Psi}) \subset Hom(\widehat{\Phi}, \widehat{\Psi}), Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}) = Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi}^{\sim}), Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}) \subset Hom^{\sim}(\Phi, \Psi), Hom^{\sim}(\widehat{\Phi}, \widehat{\Psi}) \subset Hom^{\sim}(\widehat{\Phi}^{\sim}, \widehat{\Psi})$. These morphisms and antimorphisms are not induced by morphisms $(h, h') \in Hom(\Phi, \Psi)$ and antimorphisms $(h, h')^{\sim} \in Hom^{\sim}(\Phi, \Psi)$, i.e. theorems like 1 and 6 are false.

At the same time we notice, as a conclusion, in which manner the morphisms and the antimorphisms keep the orbits, the omega limit sets, periodicity and the fixed points.

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A class of nonlocal semilinear delay evolutions

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Abstract

We state an existence result referring to a class of semilinear delay evolution equations subjected to nonlocal initial conditions.

Keywords: linear semigroups, compact perturbations, nonlocal conditions, damped wave equation.

1 Introduction

Let X be a real Banach space, let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions, let $\tau \geq 0$, $f : [\,0, T\,] \times C([\,-\tau, 0\,]; X) \to X$ a continuous and compact function having sublinear growth and let $H : C([\,-\tau, T\,]; X) \to C([\,-\tau, 0\,]; X)$ be a nonexpansive mapping. For $u \in C([\,-\tau, T\,]; X)$ and $t \in [\,0, T\,]$, we denote u_t by $u_t(s) = u(t + s)$ for each $s \in [\,-\tau, 0\,]$.

We consider the nonlocal delay differential evolution

$$\begin{cases} u'(t) = Au(t) + f(t, u_t) + g(t, u_t), & t \in [\,0, T\,], \\ u(t) = H(u)(t), & t \in [\,-\tau, 0\,]. \end{cases} \tag{1}$$

Using some metrical combined with topological fixed-point arguments, we can prove that, if f is continuous and compact, g is jointly continuous and Lipschitz with respect to its second variable and the mapping $H : C([\,-\tau, T\,]; X) \to C([\,-\tau, 0\,]; X)$ is nonexpansive in a certain sense to be made precise in due course, the problem (1) has at least one mild solution. This new theorem, inspired by those established in [3], will be proved in detail in [4] and has applications in the study of some important classes of second order hyperbolic problems subjected to nonlocal initial conditions, and not only.

2 The main result

Let $\omega > 0$. We recall that a linear operator $A : D(A) \subseteq X \to X$ is said to be the *infinitesimal generator of a C_0-semigroup of type* $(1, -\omega)$ if it generates a C_0-semigroup, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$, satisfying

$$\|S(t)x\| \leq e^{-\omega t}\|x\|$$

for each $t \in \mathbb{R}_+$ and $x \in X$. For details on delay evolutions subjected to nonlocal initial conditions and C_0-semigroups, see [1] and [2].

The main result of the paper is:

Theorem 2.1. *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of type $(1, -\omega)$, $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \in \mathbb{R}_+\}$. Let us assume that $f : [\,0, T\,] \times C([\,-\tau, 0\,]; X) \to X$ is continuous and compact and there exist $k \in \mathbb{R}_+$ and $m \in \mathbb{R}_+$ such that*

$$\|f(t, v)\| \leq k\|v\|_{C([\,-\tau,0\,];X)} + m$$

for all $(t, v) \in [\,0, T\,] \times C([\,-\tau, 0\,]; X)$, let $g : [\,0, T\,] \times C([\,-\tau, 0\,]; X) \to X$ be continuous on $[\,0, T\,] \times C([\,-\tau, 0\,]; X)$ and Lipschitz with respect to its last argument whose Lipschitz constant ℓ satisfies

$$k + \ell < \omega.$$

There exists $a \in (0, T)$ such that $H : C([\,-\tau, T\,]; X) \to C([\,-\tau, 0\,]; X)$ satisfies

$$\|H(u) - H(v)\|_{C([\,-\tau,0\,];X)} \leq \|u - v\|_{C([\,a,T\,];X)}$$

for each $u, v \in C([\,-\tau, T\,]; X)$. Then the nonlocal problem (1) *has at least one mild solution.*

3 A damped wave equation with delay

Let Ω be a nonempty bounded and open subset in $\mathbb{R}^d$, $d \geq 1$, with C^1 boundary Σ, let $Q_+ = \mathbb{R}_+ \times \Omega$, let $\tau \geq 0$, $\omega > 0$, $Q_\tau = [\,-\tau, 0\,] \times \Omega$,

$\Sigma_+ = \mathbb{R}_+ \times \Sigma$ and let us consider the non-local initial-value problem for the damped wave equation with delay:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \mathcal{L}u + h_1\left(t, \displaystyle\int_{-\tau}^{0} u(t+s,x)\,ds\right) + h_2\left(t, \left(\dfrac{\partial u}{\partial t}\right)_t\right) & \text{in } Q_+, \\ u(t,x) = 0, & \text{on } \Sigma_+, \\ u(t,x) = [H_1(u)(t)](x), \ \dfrac{\partial u}{\partial t}(t,x) = \left[H_2\left(\dfrac{\partial u}{\partial t}\right)(t)\right](x), & \text{in } Q_\tau, \end{cases} \tag{2}$$

where $[\mathcal{L}u](t,x) := \displaystyle\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(t,x) - 2\omega \frac{\partial u}{\partial t}(t,x) - \omega^2 u(t,x)$.

Theorem 3.1. *Let us assume that there exist $k \in \mathbb{R}_+$, $m \in \mathbb{R}_+$, $\gamma \in \mathbb{R}_+$ and $a \in (0,T\,]$ such that the functions h_1, h_2, H_1 and H_2 satisfy:*

(i) $h_1 : [\,0,T\,] \times \mathbb{R} \to \mathbb{R}$ *is continuous, and*

$$|h_1(t,y)| \le k|y| + m$$

for all $(t,y) \in [\,0,T\,] \times \mathbb{R}$.

(ii) $h_2 : [\,0,T\,] \times C([\,-\tau,0\,];L^2(\Omega)) \to L^2(\Omega)$ *is continuous, and*

$$\|h_2(t,y) - h_2(t,z)\|_{L^2(\Omega)} \le \gamma \|y-z\|_{C([\,-\tau,0\,];L^2(\Omega))}$$

for all $(t,y),(t,z) \in [\,0,T\,] \times C([\,-\tau,0\,];L^2(\Omega))$.

(iii) $H_1 : C([\,a,T\,];H_0^1(\Omega)) \to C([\,-\tau,0\,];H_0^1(\Omega))$, *and*

$$\|H_1(u) - H_1(\widetilde{u})\|_{C([\,-\tau,0\,];H_0^1(\Omega))} \le \|u-\widetilde{u}\|_{C([\,a,T\,];H_0^1(\Omega))}$$

for all $u, \widetilde{u} \in C([\,-\tau,0\,];H_0^1(\Omega))$.

(iv) $H_2 : C([\,a,T\,];L_0^2(\Omega)) \to C([\,-\tau,0\,];L^2(\Omega))$, *and*

$$\|H_2(v) - H_2(\widetilde{v})\|_{C([\,-\tau,0\,];L^2(\Omega))} \le \|u-v\|_{C([\,a,T\,];L^2(\Omega))}$$

for all $v, \widetilde{v} \in C([\,-\tau,0\,];L^2(\Omega))$.

(v) *With* λ_1 *the first eigenvalue of* $-\Delta$, *we have* $k+\gamma(1+\omega\lambda_1^{-1}) < \omega$.

Then, there exists at least one mild solution, u, *of the problem* (2), *satisfying* $u \in C([-\tau, T]; H_0^1(\Omega))$, *and* $\dfrac{\partial u}{\partial t} \in C([-\tau, T]); L^2(\Omega))$.

Setting $H_1(u)(t)(x) = u(t+T, x)$, $H_2\left(\dfrac{\partial u}{\partial t}\right)(t)(x) = \dfrac{\partial u}{\partial t}(t+T, x)$, and assuming that h_1 and h_2 are defined on $\mathbb{R}_+$ with respect to t, and that both are T-periodic in t, from Theorem 3.1, we deduce an existence result for T-periodic solutions to (2).

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Classification of quadratic systems possessing an invariant conic and a Darboux invariant

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Abstract

In this article we consider the family of quadratic differential systems having an invariant conic $\mathcal{C} : f(x, y) = 0$, and a Darboux invariant of the form $f(x, y)e^{st}$ with $s \in \mathbb{R} \setminus \{0\}$ (where t is the time). Applying the algebraic theory of invariants of differential equations we present a complete classification of this family of quadratic systems. First we detect necessary and sufficient conditions for an arbitrary quadratic system to be in this class. Secondly, we construct affine invariant criteria for the realization of each one of the possible phase portraits of the systems in this family.

Keywords: quadratic differential system, invariant conic, Darboux invariant, phase portrait, group action, affine invariant polynomial.

1 Introduction

We consider the family of real quadratic differential systems

$$\begin{aligned} \dot{x} &= p_0 + p_1(\tilde{a}, x, y) + p_2(\tilde{a}, x, y) \equiv p(\tilde{a}, x, y), \\ \dot{y} &= q_0 + q_1(\tilde{a}, x, y) + q_2(\tilde{a}, x, y) \equiv p(\tilde{a}, x, y) \end{aligned} \tag{1}$$

with $\max(\deg(p), \deg(q)) = 2$ and

$$\begin{aligned} &p_0 = a, \quad p_1(\tilde{a}, x, y) = cx + dy, \quad p_2(\tilde{a}, x, y) = gx^2 + 2hxy + ky^2, \\ &q_0 = b, \quad q_1(\tilde{a}, x, y) = ex + fy, \quad q_2(\tilde{a}, x, y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

Here we denote by $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ the 12-tuple of the coefficients of systems (1).

In [3] the family of systems (1) possessing an invariant conic and a Darboux invariant is considered.

Let Ω be an open and dense subset of $\mathbb{R}^2$. According to [3] *an invariant* of systems (1) in Ω is a nonconstant C^1 function I in the variables x, y and t such that $I(x(t), y(t), t)$ is constant on all solution curves $(x(t), y(t))$ of system (1) contained in Ω, i.e. $\frac{\partial I}{\partial x}p + \frac{\partial I}{\partial y}q + \frac{\partial I}{\partial t} = 0$, for all $(x, y) \in \Omega$.

For $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is *an invariant algebraic curve* of system (1) if there exists $K \in \mathbb{C}[x, y]$ such that $P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf$.

In paper [3, Subsection 2.2] it is given the definition of a Darboux invariant of the family of polynomials differential systems. We consider here a particular case of a Darboux invariant, namely an invariant of the form $I(x, y, t) = f(x, y)^\lambda e^{st}$ with $\lambda, s \in \mathbb{R}$, $s \neq 0$, where $f(x, y) = 0$ is an invariant algebraic curve.

According to [3] the following statements hold:

Lemma [3, Theorems 2 and 8].

(i) *The only planar quadratic systems which admit a non-degenerate invariant conic $\mathcal{C} : f(x, y) = 0$ and a Darboux invariant of the form $f(x, y)^\lambda e^{st}$ with $\lambda, s \in \mathbb{R}$ and $s \neq 0$ are those for which $\mathcal{C}$ is a parabola.*

(ii) *Quadratic systems having the invariant parabola $y = x^2$ and a Darboux invariant of the form $(y - x^2)^\lambda e^{st}$ with $\lambda, s \in \mathbb{R}$ and $s \neq 0$ via an affine transformation can be brought to the form*

$$\dot{x} = px + qy + r, \quad \dot{y} = c(y - x^2) + 2x(px + qy + r)$$

where c, p, q and r are real parameters. Moreover its Darboux invariant is $I(x, y, t) = (y - x^2)e^{-ct}$.

2 Main results

In the next theorem we use the invariant polynomials μ_i $(i = 0, 1, \ldots, 4)$, $\widetilde{M}$, η, $\widetilde{K}$, $\widetilde{R}$, $\mathbf{D}$, K_1 and K_3 which were constructed

earlier and could be found, for example, in [1] and [2]). We also use the invariant polynomials V_1, V_2 and V_3 constructed here as follows:
$V_1 = A_{17} - A_{18}$, $V_2 = \frac{\partial C_2}{\partial x}\frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial y}\frac{\partial C_1}{\partial x}$, $V_3 = 4\frac{\partial C_2}{\partial x}\frac{\partial C_0}{\partial y} - 4\frac{\partial C_2}{\partial y}\frac{\partial C_0}{\partial x} - 3C_1D_1$, where A_{17} and A_{18} are the affine invariants given in [1] and
$C_i = yp_i - xq_i, (i = 0, 1, 2)$, $D_1 = \frac{\partial}{\partial x}p_1 + \frac{\partial}{\partial y}q_1$.

Taking into consideration the statements of the above lemma we prove the following theorem:
Main Theorem. (*A*) *A non-degenerate quadratic differential system in the class* **QS** *(i.e.* $\sum_{i=0}^{4}\mu_i^2 \neq 0$*) possesses an irreducible invariant conic* $f(x,y) = 0$ *and a Darboux invariant of the form* $f(x,y)^{\lambda}e^{st}$ *with* $\lambda, s \in \mathbb{R}$ *and* $s \neq 0$ *if and only if* $\eta = \widetilde{K} = \widetilde{R} = 0$ *and one of the following sets of conditions holds:*

$$(i)\ \widetilde{M}\mu_2 \neq 0, V_1 = 0;\ \ (ii)\ \widetilde{M} = \mu_2 = 0, K_3 \neq 0\ and\ (V_2 \neq 0) \vee (V_2 = V_3 = 0).$$

Moreover this system has an one-parameter family of invariant parabolas if an only if $M = \mu_2 = V_2 = V_3 = 0$*, whereas in all other cases the invariant parabola is unique.*

(*B*) *Assume that a non-degenerate system in* **QS** *possesses an invariant conic (which is parabola) and a Darboux invariant, i.e. the conditions provided by the statement* **(*A*)** *are satisfied. Then the phase portrait of this system corresponds to one of the given in Figure 1 if and only if the corresponding additional conditions are satisfied as follows:*

Port.1 ⇔ $\widetilde{M} \neq 0$, $\mathbf{D} < 0$; *Port.2* ⇔ $\widetilde{M} \neq 0$, $\mathbf{D} > 0$;
Port.3 ⇔ $\widetilde{M} \neq 0$, $\mathbf{D} = 0$; *Port.4* ⇔ $\widetilde{M} = 0$, $\mu_3K_1 < 0$;
Port.5 ⇔ $\widetilde{M} = 0$, $\mu_3K_1 > 0$, $K_3 < 0$;
Port.6 ⇔ $\widetilde{M} = 0$, $\mu_3K_1 > 0$, $K_3 = 0$;
Port.7 ⇔ $\widetilde{M} = 0$, $\mu_3K_1 = 0$.

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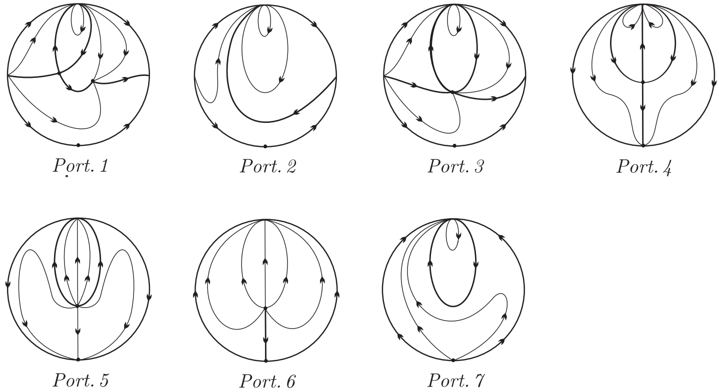

Figure 1. **The phase portraits**

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Absolute Matrix Summability Factors of Involving quasi-f-power Increasing Sequence

Şebnem Yildiz

Abstract

In this work, a new theorem dealing with absolute Riesz summability factors of infinite series has been proved by taking normal matrix and using quasi-f-power increasing sequence.

Keywords: Riesz mean, absolute matrix summability, summability factors, infinite series, Hölder inequality, Minkowski inequality.

1 Introduction

A positive sequence (b_n) is said to be *almost increasing* if there exists a positive increasing sequence (b_n) and two positive constants A and B such that $Ab_n \leq a_n \leq Bb_n$ (see [1]). A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. A positive sequence $X = X_n$ is said to be a *quasi-f-power* increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \left\{n^{\delta}(logn)^{\sigma}, \sigma \geq 0, 0 < \delta < 1\right\}$ (see [7]). Let (p_n) be a sequence of positive number such that $P_n = \sum_{v=0}^{\infty} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)$. The sequence-to-sequence transformation $t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$ defines the sequence (t_n) of the Riesz mean or simply the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where $A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, ...$ The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{2}$$

where (θ_n) is any sequence of positive constants and $\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s)$.

2 The Known Result

Bor has obtained the following result concerning $|\bar{N}, p_n|_k$ summability factors of infinite series in the following form.

Theorem 1.[3] *Let $(\lambda_n) \in \mathcal{BV}$, and let (X_n) be a quasi-f-power increasing sequence for some δ $(0 < \delta < 1)$ and $\sigma \geq 0$. Let (β_n) and (λ_n) be sequences such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{3}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{4}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{5}$$

$$|\lambda_n|X_n = O(1), \tag{6}$$

and let (p_n) be a sequence such that

$$P_n = O(np_n), \tag{7}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{8}$$

If

$$\sum_{v=1}^{n} \frac{|t_v|^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{9}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3 The Main Result

The aim of this paper is to generalize Theorem 1 for $|A, \theta_n|_k$ summability method.

Theorem 2. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ \ n = 0, 1, ..., \tag{10}$$

$$a_{n-1,v} \geq a_{nv}, \ \ \text{for} \ \ n \geq v + 1, \tag{11}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{12}$$

$$\hat{a}_{n,v+1} = O(v|\bar{\Delta} a_{nv}|). \tag{13}$$

Let $(\lambda_n) \in \mathcal{BV}$, and let (X_n) be a quasi-f-power increasing sequence for some δ $(0 < \delta < 1)$ and $\sigma \geq 0$. If the conditions (3)-(8) of Theorem 1 are satisfied and $(\theta_n a_{nn})$ is a non-increasing sequence satisfying

$$\sum_{v=1}^{n} (\theta_v a_{vv})^{k-1} \frac{|t_v|^k}{v} = O(X_n), \quad as \quad n \to \infty. \tag{14}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|A, \theta_n|_k$, $k \geq 1$.

Lemma 1[5] Under the conditions on (X_n), (β_n), and (λ_n) as expressed in the statement of Theorem 1, we have the following:

$$n X_n \beta_n = O(1), \tag{15}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{16}$$

Lemma 2 [4] If the conditions (7)-(8) of Theorem 1 are satisfied, then $\Delta \left(\frac{P_n}{n^2 p_n} \right) = O\left(\frac{1}{n^2}\right)$.

4 Conclusion

If we take A as the matrix of weighted mean with $\theta_n = \frac{P_n}{p_n}$ in Theorem 2, we have Theorem 1 dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.

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A New Generalization on the Absolute Riesz Summability

Şebnem Yildiz

Abstract

In this work, a new theorem dealing with absolute Riesz summability factors of infinite series has been proved by using quasi increasing sequence.
Keywords: Riesz mean, absolute matrix summability, summability factors, infinite series, Hölder inequality, Minkowski inequality.

1 Introduction

A positive sequence (a_n) is said to be *almost increasing* if there exists a positive increasing sequence (b_n) and two positive constants A and B such that $Ab_n \leq a_n \leq Bb_n$ (see [1]). A sequence (a_n) is almost increasing if and only if it is *quasi increasing*, that is if there exists a constant $K = K(a_n) \geq 1$ such that $Ka_n \geq a_m \geq 0$ holds for all $n \geq m$. Let (p_n) be a sequence of positive number such that $P_n = \sum_{v=0}^{\infty} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)$. The sequence-to-sequence transformation $t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$ defines the sequence (t_n) of the Riesz mean or simply the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s=(s_n)$ to $As=(A_n(s))$, where $A_n(s)=\sum_{v=0}^{n} a_{nv}s_v, \quad n=0,1,...$ The series $\sum a_n$ is said to be summable $|A,\theta_n|_k$, $k\geq 1$, if (see [3])

$$\sum_{n=1}^{\infty}\theta_n^{k-1}\left|\bar{\Delta}A_n(s)\right|^k<\infty, \tag{2}$$

where (θ_n) is any sequence of positive constants and

$$\bar{\Delta}A_n(s)=A_n(s)-A_{n-1}(s). \tag{3}$$

2 The Known Result

Sulaiman has obtained the following result concerning $|\bar{N},p_n|_k$ summability factors of infinite series in the following form.

Theorem 1.[4] *Let $(\lambda_n)\to 0$. Suppose there exists a positive quasi increasing sequence (X_n) such that*

$$\sum_{n=1}^{\infty}\frac{1}{n}X_n|\lambda_n|<\infty, \tag{4}$$

$$\sum_{n=1}^{m}\frac{1}{n}\frac{|t_n|^k}{X_n^{k-1}}=O(X_m) \quad as \quad m\to\infty, \tag{5}$$

$$\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{|t_n|^k}{X_n^{k-1}}=O(X_m) \quad as \quad m\to\infty, \tag{6}$$

$$\sum_{n=1}^{\infty}X_n|\Delta\lambda_n|<\infty, \tag{7}$$

$$\sum_{n=1}^{\infty}nX_n|\Delta|\Delta\lambda_n||<\infty, \tag{8}$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.

3 The Main Result

The aim of this paper is to generalize Theorem 1 for $|A, \theta_n|_k$ summability method.

Theorem 2. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ n = 0, 1, ..., \tag{9}$$

$$a_{n-1,v} \geq a_{nv}, \ \text{for} \quad n \geq v + 1. \tag{10}$$

$$a_{nn} = O(\frac{p_n}{P_n}) \tag{11}$$

and let $(\theta_n a_{nn})$ be a non-increasing sequence. If (X_n) satisfy the conditions (4)-(5) and (7)-(8) of Theorem 1 and the following conditions holds by (θ_n)

$$\sum_{n=v+1}^{\infty} \theta_n^{k-1} \hat{a}_{nv}^k < \infty, \tag{12}$$

$$\sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} a_{nn} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m), \quad as \quad m \to \infty \tag{13}$$

then the series $\sum a_n \lambda_n$ is summable $|A, \theta_n|_k$, $k \geq 1$.

Lemma 1.[4] *Let (X_n) be a positive quasi increasing sequence and let $\lambda_n \to 0$. If the conditions (7)-(8) of Theorem 1 are satisfied, then*

$$X_n |\lambda_n| < \infty, \tag{14}$$

$$n X_n |\Delta \lambda_n| < \infty. \tag{15}$$

4 Conclusion

If we take A as the matrix of weighted mean with $\theta_n = \frac{P_n}{p_n}$ in Theorem 2, we have Theorem 1 dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.

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Systems of differential equations approximating the Lorenz system

Biljana Zlatanovska, Donco Dimovski

Abstract

By using modified Lorenz system from [1] as the system of differential equations of seventh order which approximated the Lorenz system, we obtained four new systems of differential equations of third, fourth, fifth and sixth order. Every new system of differential equations is obtained using the solutions of the third differential equation from the modified Lorenz system. The third differential equation of modified Lorenz system is homogeneous linear differential equation of fifth order with constant coefficients which can be solved. By computer simulations we compare the local behavior of modified systems of differential equations with the global behavior of the Lorenz system.

Keywords: Lorenz system, system of differential equations, modified Lorenz system, computer simulations.

1 Introduction

In [2] and [3] we have used power series combined with difference equations to find local approximations to the solutions of the Lorenz system of differential equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(r - z) - y \\ \dot{z} &= xy - bz \end{aligned} \tag{1}$$

with parameters σ, r and b. For initial values $a_0=x(0)$, $b_0=y(0)$, $c_0=z(0)$. We assume that the solutions x(t), y(t), z(t) of the system (1) are expanded as Maclaurin series with the coefficients a_n, b_n, c_n.

By [2], [3] and [1] after mathematical transformations with the initial values $a_0=x(0)$, $b_0=y(0)$, $c_0=z(0)$, $c_p = \overset{(p)}{z}(0)$, $p \in \{1,2,3,4\}$ and for $A=1+\sigma+b$, $B=\sigma(r-c_0)-a_0^2$, $C=\sigma a_0 b_0$, $D= -\sigma^2 b_0^2$, it was obtained modified

Lorenz system ([1]),

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(r-z)-y\\ \overset{(5)}{z} &= -(A+b)\overset{(4)}{z}+(B-Ab)\overset{(3)}{z}-(C-Bb)\overset{(2)}{z}+(D-Cb)\dot{z}-Dbz\end{aligned} \tag{2}$$

2 Systems of differential equations

The third equation of the system (2) is homogenous linear differential equation of fifth order with constant coefficients and its characteristic equation has solutions $\lambda_1 = -b$, $\lambda_{2/3/4/5}=\lambda(A,B,C,D,b)$. Let, we suppose that all solutions of characteristic equation λ_i, i=1,2,3,4,5 are real solutions.

For the solutions $\lambda_{1/2/3/4/5}$ the system (2) of seventh order can be transformed in the following systems of differential equations (SDE):

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(r-z)-y\\ \dot{z}-\lambda_1 z &= 0\end{aligned} \tag{3}$$

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(r-z)-y\\ \overset{(2)}{z}-(\lambda_1+\lambda_2)\dot{z}+\lambda_1\lambda_2 z &= 0\end{aligned} \tag{4}$$

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(r-z)-y\\ \overset{(3)}{z}-(\lambda_1+\lambda_2+\lambda_3)\overset{(2)}{z}+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\dot{z}-\lambda_1\lambda_2\lambda_3 z &= 0\end{aligned} \tag{5}$$

$$\begin{aligned}&\dot{x} = \sigma(y-x)\\ &\dot{y} = x(r-z)-y\\ &\overset{(4)}{z}-(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\overset{(3)}{z}+\\ &(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\overset{(2)}{z}-\\ &(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4)\dot{z}+\lambda_1\lambda_2\lambda_3\lambda_4 z = 0\end{aligned} \tag{6}$$

with the initial values a_0=x(0),b_0=y(0),c_0=z(0), $\dot{z}(0)=c_1$, $\overset{(2)}{z}(0)=c_2$, $\overset{(3)}{z}(0)=c_3$.

3 Computer simulations for the SDE

In this section, we will look via computer simulations the local behavior for the SDE (2), (3), (4), (5), (6) and we will compare with global behavior of the Lorenz system,(fig.1). For given parameters σ, r, b, the procedure for looking at the local behavior of the SDE is the same as in [4], fig.2.

Example: For the parameters σ=2, r=31, b=1 and the initial values a_0=-3, b_0=1, c_0=-5 and λ_1=-1, λ_2≈-0.301, λ_3≈-10.147, λ_4≈0.210.

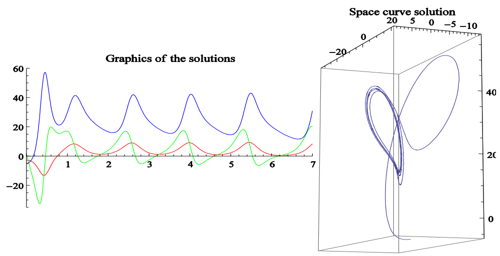
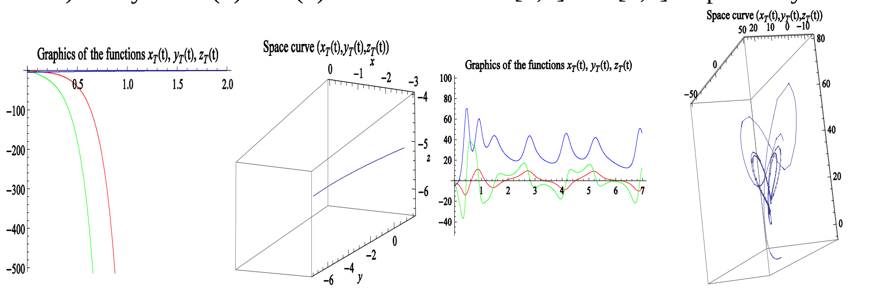
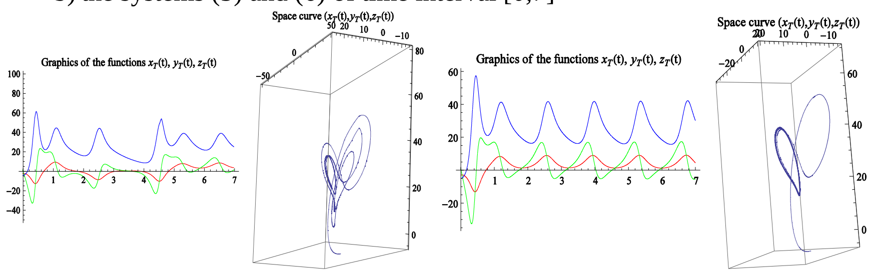


Figure 1: Results obtained by Mathematica for the Lorenz system (1) of time interval [0,7]

a) the systems (3) and (4) of time intervals [0,2] and [0,7] respectively



b) the systems (5) and (6) of time interval [0,7]



c) the system (2) of time interval [0,7]

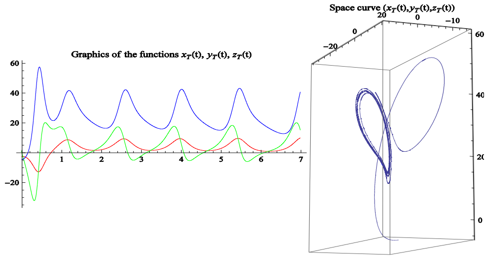


Figure 2: The solutions $x_T(t)$, $y_T(t)$, $z_T(t)$ for the systems (2), (3), (4), (5) and (6) with time step T=0.05

4 Conclusion

The local behavior of the system (2) is closest to the behavior of the Lorenz system for a small time step.

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Section 4

Applied Mathematics

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Maximum nontrivial convex cover of a tree

Radu Buzatu

Abstract

The nontrivial convex p-cover problem of a tree is studied. We propose the recursive formula that determines the maximum nontrivial convex cover number of a tree.

Keywords: convexity, nontrivial convex cover, tree.

1 Introduction

A vertex set S of a graph G is called *convex* if all vertices of every shortest path between two of its vertices are in S [1]. Generally, the concept of *convex p-cover* of a graph is introduced in [2] and is examined in [2, 3]. We defined a nontrivial convex p-cover of a graph as a special case of general convex p-cover in [3]. A family of sets $\mathcal{P}_p(G)$ is called a *nontrivial convex p-cover* of a graph G if the following conditions hold:

1) every set of $\mathcal{P}_p(G)$ is convex in G;
2) every set S of $\mathcal{P}_p(G)$ satisfies inequalities: $3 \leq |S| \leq |X(G)| - 1$;
3) $X(G) = \bigcup_{Y \in \mathcal{P}_p(G)} Y$;
4) $Y \not\subseteq \bigcup_{\substack{Z \in \mathcal{P}_p(G) \\ Z \neq Y}} Z$ for every $Y \in \mathcal{P}_p(G)$;
5) $|\mathcal{P}_p(G)| = p$.

Particularly, we showed that it is NP-complete to decide whether a graph has a nontrivial convex p-cover for a fixed $p \geq 2$ [3]. Nontrivial convex p-covers for some classes of graphs are studied in [7, 8]. The most consistent results are obtained for trees [4, 5, 6]. In the present paper we continue our research on nontrivial convex p-cover problem of a tree.

2 Main Results

Recall that a vertex $x \in X(G)$ is called *resident* in $\boldsymbol{\mathcal{P}}_p(G)$ if x belongs to only one set of $\boldsymbol{\mathcal{P}}_p(G)$ [3]. The greatest $p \geq 2$ for which a graph G has a nontrivial convex p-cover is said to be the *maximum nontrivial convex cover number* $\varphi_{cn}^{max}(G)$ [4].

Let T be a tree on n vertices and let $C(T)$ be a set of terminal vertices of T, $p = |C(T)|$. An important result is given by the following lemma.

Lemma 1. *If $n \geq 4$, then there exists a maximum nontrivial convex cover $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$ such that every set $S \in \boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$ contains a path $L = [x, y, z]$, where x is a resident vertex in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$.*

Suppose that $diam(T) \geq 4$, then we define the set:

$$N(T) = X(T) \backslash \left(C(T) \cup \bigcup_{y \in C(T)} \Gamma(y) \right).$$

The set $N(T)$ is empty if and only if every nonterminal vertex of T is adjacent to at least one terminal vertex of T, but in this case, according to [4], we get $\varphi_{cn}^{max}(T) = p$. Let x be a vertex of $N(T)$. Since x is an articulation vertex, through the elimination of x from T we obtain $|\Gamma(x)|$ connected components T_x^y, $y \in \Gamma(x)$. For every vertex $y \in \Gamma(x)$ we get the family of subtrees:

$$\boldsymbol{\mathcal{V}}_x^y(T) = {}^*T_x^y \cup \bigcup_{z \in \Gamma(x) \backslash y} T_x^z,$$

where ${}^*T_x^y$ is a subtree of T obtained by adding x to T_x^y such that x is adjacent to y.

Finally, we get the family of subfamilies of subtrees:

$$\boldsymbol{\mathcal{V}}_x(T) = \bigcup_{y \in \Gamma(x)} \boldsymbol{\mathcal{V}}_x^y(T).$$

For the sake of estimation of the number $\varphi_{cn}^{max}(T)$, we consider that if $0 \leq n \leq 2$, then $\varphi_{cn}^{max}(T) = 0$, and if $n = 3$, then $\varphi_{cn}^{max}(T) = 1$. Combining Lemma 1 with results from [4, 5, 6], we obtain the recursive formula, reflected in Theorems 1 and 2, that determines the maximum nontrivial convex cover number $\varphi_{cn}^{max}(T)$.

Theorem 1. *If $diam(T) \leq 5$ or $diam(T) \geq 6$ and $N(T) = \varnothing$, then the following relation holds:*

$$\varphi_{cn}^{max}(T) = \begin{cases} p, & \text{if } 3 \leq diam(T) \leq 5 \text{ or} \\ & diam(T) \geq 6 \text{ and } N(T) = \varnothing; \\ p\text{-}1, & \text{if } diam(T) = 2; \\ 0, & \text{if } 0 \leq diam(T) \leq 1. \end{cases}$$

Theorem 2. *If $diam(T) \geq 6$ and $N(T) \neq \varnothing$, then the following relation holds:*

$$\varphi_{cn}^{max}(T) = max\left\{p, max_{x\in N(T)}\left\{max_{y\in\Gamma(x)}\left\{\sum_{H\in\boldsymbol{\mathcal{V}}_x^y(T)} \varphi_{cn}^{max}(H)\right\}\right\}\right\}.$$

3 Conclusion

In this paper we propose the recursive formula that establishes the maximum nontrivial convex cover number of a tree, based on which an efficient algorithm that determines whether a tree has a nontrivial convex p-cover for a fixed $p \geq 2$ can be developed. Taking into account our previous results [4, 5, 6] together with these new findings we arrive to the conclusion that the nontrivial convex p-cover problem of a tree is almost completely solved.

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Algorithm for the localization of singularities of functions defined on closed contours

Maria Capcelea, Titu Capcelea

Abstract

A numerical algorithm for locating polar singularities of functions defined on a discrete set of points of a simple closed contour in the complex plane is examined. The algorithm uses the Faber-Padé approximation of the function and the fact that the zeros of its denominator give us approximations of the poles of function. The numerical performance of the algorithm is being analyzed on test issues.

Keywords: Padé algorithm, singular points, closed contour.

1 Problem Formulation

Methods for solving differential and integral equations whose solutions are meromorphic functions usually start from the premise that locations of the singularities of functions are known. In this paper we examine the following problem of locating singularities (discontinuity points and poles, not the essential singularities) of functions defined on contours in complex plane.

Let $\Omega^+ \subset \mathbb{C}$ be a simply connected domain bounded by a piecewise smooth closed curve Γ. We consider that the point $z = 0 \in \Omega^+$. By the Riemann mapping theorem there exists a conformal map $z = \psi(w)$ of $D^- := \{w \in \mathbb{C} \mid |w| > 1\}$ onto $\Omega^- := \overline{\mathbb{C}} \setminus \{\Omega^+ \cup \Gamma\}$ such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. The function $\psi(w)$ transforms the circle $\Gamma_0 := \{w \in \mathbb{C} \mid |w| = 1\}$ onto Γ.

Let $f(z)$ be a meromorphic function in the finite domain $\Omega \supset \Omega^+ \cup \Gamma$ that is analytic in Ω^+ (we will denote this fact by $f \in H(\Omega^+)$). Considering that the function $f(z)$ is defined by a finite set of values $f_k = f(z_k)$ in the points $\{z_k\}$, $z_k = \psi(w_k) \in \Gamma$, $w_k \in \Gamma_0$, we aim to find the singularities of f on Γ.

2 The theory underlying the algorithm

According to the theory of analytic continuation of functions of a complex variable, the properties of a function analytic at a point are contained in its Taylor series expansion at that point. We know that the Padé approximants perform an analytic continuation of the series outside its domain of convergence and can be used effectively in determining information about the singularity structure of a function from its Taylor series coefficients [1].

The problem of recovering the meromorphic function F in the disk $D_m(F)$, where F has m poles taking into account their multiplicities, is solved on the basis of the theorem of de Montessus de Ballore [1]. According to Montessus's theorem, the poles of the sequence of Padé approximants to the function F converge to the poles of F on $D_m(F)$.

In the formulated problem in order to apply the classical result of Montessus's theorem we used the properties of the Faber transform [2,3] and of the conformal map $z = \psi(w)$. Since the function f is analytic on simply connected domain Ω^+ with boundary Γ, the Faber series expansion [2,3] is used to represent the function f instead of the Taylor series that is defined on the disk.

The Faber transform T associates to $F \in H(D^+)$ the Faber series expansion of the function $f \in H(\Omega^+)$, $f(z) = \sum_{k=0}^{\infty} c_k F_k(z)$, $z \in \Omega^+$, where $F_k(z)$ is the Faber polynomial of degree k [2,3]. The coefficients of the expansion are defined by the formula $c_k = \frac{1}{2\pi i} \int_{\Gamma_0} f(\psi(w))\, w^{-(k+1)} dw$. For $f \in H(\Omega^+)$ there exists an $F \in H(D^+)$ such that $f = T(F)$.

An important property of the Faber operator is that it induces a bijective correspondence between the set of rational functions with poles on $D^{-} \cup \Gamma_0$ and the set of rational functions with poles on $\Omega^{-} \cup \Gamma$. Moreover, this transformation keeps intact the number of poles and their multiplicities [3]. The poles of $T(R)$ are obtained as images under the Riemann function ψ of the poles of rational function R [3].

Based on the Montessus's theorem [1] it can be shown for the meromorphic function f with M poles on $\Omega^{-} \cup \Gamma$ that for sufficiently large N, the Padé approximants $r_{(N,M)}$ to f have M poles. As $N \to \infty$ the sequence $r_{(N,M)}$ converges to f uniformly inside the domain Ω' obtained from Ω by deleting the poles of f and the poles of the sequence $r_{(N,M)}$ tend to the poles of f. Each pole of f attracts a number of poles of $r_{(N,M)}$ equal to its multiplicity.

3 An algorithm for the localization of singular points on Γ

For the classical Padé approximant $R_{(N,M)}(w)$ to the function $F(w)$ we determine the poles that belong to Γ_0. Next, by using the properties of the Faber transform, the singular points on Γ of the Faber-Padé approximation are located. We perform the following steps:

1. We compute the coefficients q_j, $j = 1, ..., M$ of the polynomial $Q_M(w)$ from the Padé approximation $R_{(N,M)}(w) = P_N(w)/Q_M(w)$ to $F(w)$. If we have $P_N(w) = \sum_{k=0}^{N} p_k w^k$, $Q_M(w) = \sum_{j=0}^{M} q_j w^j$, $q_0 = 1$, then the coefficients q_j, $j = 1, ..., M$ are determined as a solution of the system of linear equations (abbreviated SLE):

$$\sum_{j=1}^{M} c_{k-j} q_j = -c_k,\ k = N+1, ..., N+M,$$

where c_j are the Faber coefficients that coincide with the Taylor coefficients for the function $F(w) = \sum_{j=0}^{\infty} c_j w^j$.

Taking into account that the function f is defined by its values on the boundary Γ, we will compute approximations to c_j based on m - point trapezoidal rule for the contour integral. To avoid the situation when the singularities of f on Γ affect the accuracy of the approximation, we will approximate the integral on Γ_0 that defines c_k by the integral on the perturbed circle $\Gamma_0^\rho := \{ w \in \mathrm{C} \mid |w| = \rho \}$. Here we have $\rho = 1 - \varepsilon > 0$ and $\varepsilon > 0$ is given arbitrarily small positive number, for example, $\varepsilon = 0.01$. If $\theta_k \in [0, 2\pi]$, $k = 0, 1, ..., m$ are the polar angles corresponding to the points z_k, $k = 0, 1, ..., m$ on Γ, then according to m - point trapezoidal rule we have

$$c_j \approx \tilde{c}_j^{(m)} := \frac{1}{4\pi\rho^j} \sum_{k=1}^{m} (\theta_k - \theta_{k-1}) \left(f_{k-1} e^{-ij\theta_{k-1}} + f_k e^{-ij\theta_k} \right),$$

where the values $f_k = f\left(\psi\left(\rho e^{i\theta_k}\right)\right)$, $k = 0, ..., m-1$ are initially given.

2. We find the zeros of the polynomial $Q_M(w) = \sum_{j=0}^{M} q_j w^j$, $w \in \Gamma_0$, where q_j, $j = 1, ..., M$ is the solution of SLE. Next we find the zeros of the polynomial $q_M(z)$, $z \in \Gamma$, as images under ψ of the zeros of $Q_M(w)$. The obtained values are the candidates for the desired singularities.

The numerical performance of the algorithm is analyzed. Also, we discuss how to eliminate the spurious poles of the Padé approximants.

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d^m-convex functions in the complex of multi-ary relations

Sergiu Cataranciuc, Galina Braguţa

Abstract

In the present work the notions of d^m-convexity and d^m-convex function are defined. Some properties of these functions are mentioned. We study the complexes of multi-ary relations for which the median function is d^m-convex.

Keywords: complex of multi-ary relations, m-dimensional chain, d^m-convex set, d^m-convex function.

1 Introduction

Solving of many optimization problems consists in examination of some functions on discrete structures. In this context, convex functions have a special role. The functions property of being convex guarantees elaboration of efficient methods that determine the optimal solution of problem. For these reasons, the determination of conditions that ensure the convexity of special functions in a complex of multi-ary relations is important. Such functions frequently occur in process of studying location problems and are known as median-functions.

Let $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ be a complex of multi-ary relations, determined by a finite set of elements $X = \{x_1, x_2, ..., x_p\}$. The complex of relations $\Re^{n+1}$ was thoroughly defined and studied in the work [1]. According to the definition, R^m, $1 \leq m \leq n+1$ represents a subset of the Cartesian product of rank m of the set X and is not empty.

We choose two elements $r^k \in R^k$, $r^q \in R^q$, $1 \leq k, q < n+1$. We generalize the concept of chain in $\Re^{n+1}$ used in works [2], [3].

The sequence of the elements $r^m_{t_1}$,$r^m_{t_2}$,...,$r^m_{t_s} \in R^m$, $max\{k, q\} < m \leq n + 1$, with following properties:

a) $r^k \subset r^m_{t_1}$;

b) $r^q \subset r^m_{t_s}$;

c) $r^m_{t_p} \cap r^m_{t_{p+1}} \in R^l$, $1 \leq l < m$, for any p, $1 \leq p \leq s - 1$,

is called m-dimensional chain with the extremities in $r^k \in R^k$, $r^q \in R^q$ and is denoted by $L^m(r^k, r^q) = [r^m_{t_1}, r^m_{t_2}, ..., r^m_{t_s}]$.

The number s is called the length of the chain $L^m(r^k, r^q)$. The minimal length of m-dimensional chains that connect elements $r^k \in R^k$ and $r^q \in R^q$ is called m-distance between these elements and is denoted by $d^m(r^k, r^q)$. If between two elements $r^k \in R^k$ and $r^q \in R^q$ there does not exist m-dimensional chain, then it is considered that $d^m(r^k, r^q) = +\infty$.

2 d^m-convex functions

We mention that m-distance is defined on the elements of the set $R^1 \cup R^2 \cup ... \cup R^{m-1}$, $m \geq 2$, and possesses metric properties:

a) $d^m(r^k, r^q) \geq 0$, for any two elements $r^k \in R^k$ and $r^q \in R^q$, and $d^m(r^k, r^q) = 0$ if and only if $r^k \in r^q$;

b) $d^m(r^k, r^q) = d^m(r^q, r^k)$, for any two elements $r^k \in R^k$, $r^q \in R^q$;

c) $d^m(r^k, r^q) \leq d^m(r^k, r^l) + d^m(r^l, r^q)$, for any three elements $r^k \in R^k$, $r^l \in R^l$, $r^q \in R^q$ $(k, l, q < m)$.

Definition 1. *A complex of multi-ary relations $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ is m-conex if for any two elements $r^k \in R^k$ and $r^q \in R^q$, $1 \leq k, q < m$ an m-dimensional chain $L^m(r^k, r^q)$ exists.*

Theorem 1. *If $\Re^{n+1}$ is an m-dimensional complex of multi-ary relations, then it is also h-dimensional for any h, $2 \leq h < m$.*

Let $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ be a complex of multi-ary relations and $\Re^{m-1} = (R^1, R^2, ..., R^{m-1})$, $2 \leq m \leq n + 1$ be a subcomplex from $\Re^{n+1}$. On elements of $\Re^{n+1}$ we define a function with values in the set of real numbers $f : \Re^{n+1} \to R$.

Similarly to those mentioned in the paper [4], this function is d^m-convex if for any three elements r^k, r^q and r^h from subcomplex $\Re^{m-1}$ with the property $d^m(r^k, r^q) = d^m(r^k, r^h) + d^m(r^h, r^q)$ in $\Re^{n+1}$, the following inequality holds: $f^m(r^h) \leq \frac{d^m(r^h,r^q)}{d^m(r^k,r^q)} f^m(r^k) + \frac{d^m(r^k,r^h)}{d^m(r^k,r^q)} f^m(r^q)$. Of course, the function is defined if the right side of the inequality exists.

Theorem 2. *If f^m is a d^m-convex function defined on the complex of multi-ary relations $\Re^{n+1}$, then every local extremum of this function coincides with the global.*

Theorem 3. *The sum of two d^m-convex functions defined on the complex of multi-ary relations $\Re^{n+1}$ is a d^m-convex function in $\Re^{n+1}$.*

Definition 2. *The set $A \subset R^1 \cup R^2 \cup ... \cup R^{m-1}$ is called m-convex in $\Re^{n+1}$ if every m-dimensional chain that connects two elements of A contains only elements from A.*

Theorem 4. *If f is a d^m-convex function in $\Re^{n+1}$ and α is a real number, then the set $\{r \in A = R^1 \cup R^2 \cup ... \cup R^{m-1} : f(r) \leq \alpha\}$ is d^m-convex in the complex of multi-ary relations $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$.*

3 Median function in the complex of multi-ary relations

Median functions are used to solve services centre location problems. A location problem on a complex of multi-ary relations consists in minimisation of the function $F : A = R^1 \cup R^2 \cup ... \cup R^{m-1} \to R$ of the following type: $F^m(r) = \sum\limits_{z \in A} d^m(r, z)$.

The solving of such problems is quite complicated, because, in general case, the function $F^m(r)$ does not have any properties that would facilitate the determinations of the extremum. However, for some special complexes, the situation becomes quite favourable.

If $r^k = r^q$ we say that the chain $L^m(r^k, r^q)$ is a m-dimensional cycle.

Theorem 5. *If the complex of multi-ary relations* $\Re^{n+1}$ *does not contain* h*-dimensional cycles,* $h \leq m$*, then the median function* $F^m(r)$ *is* d^m*-convex.*

Theorem 6. *If* $F^m(r)$ *is a* d^m*-convex median function, then all extremal points of the function* $F^m(r)$ *generate a convex subcomplex in the complex of multi-ary relations.*

4 Conclusion

In this paper there are presented the results about the study of convex functions properties, based on the generalization of the notion of metric convexity, known from graphs theory. The obtained results contribute to the development of the convexity theory in complexes of multi-ary relations.

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The Explicit Solution and Computer Modeling of the Spatial Electrodynamic Problem

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Abstract

The specific case of the differential Maxwell system in the rectangular coordinates is considered as the basic mathematical model of electromagnetic wave propagation in the spatial guided structures. The general wave PDE (partial differential equation) is got including all scalar components of the electromagnetic field vector intensities. Equivalence of this PDE to the original Maxwell statement is proved. Mathematical simulation of the electrodynamic phenomenon is done using boundary value problem regarding the aforesaid wave equation. This problem is solved explicitly. The obtained exact formulae are used for numerical implementation and computer modeling of the considered engineering process.

Keywords: differential Maxwell system, general wave equation.

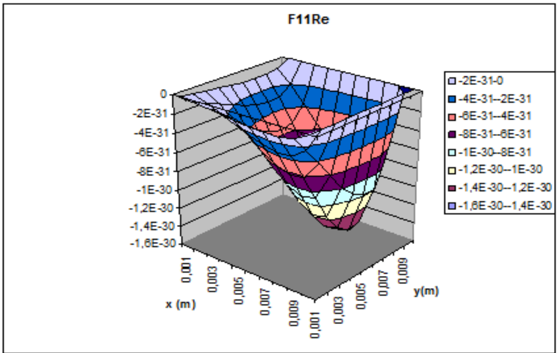
1 Preliminaries

The explicit solution of the following differential Maxwell system in the Cartesian coordinates is proposed here using analytic technique of [1]:

$$\begin{cases} \mathbf{rot}\ \vec{E} = -\partial_0 \vec{B},\ \mathbf{rot}\ \vec{H} = \partial_0 \vec{D} + \vec{i}\,;\ \vec{i} = \sigma \vec{E}; \\ \operatorname{div} \vec{D} = \rho,\ \vec{D} = \varepsilon \vec{E}; \operatorname{div} \vec{B} = 0,\ \vec{B} = \mu \vec{H}. \end{cases} \tag{1}$$

In Sys. (1): $\vec{E}, \vec{H}(x, y, z, t)$ are the unknown electromagnetic field vector intensities and harmonic regarding the time argument; $\vec{D}, \vec{B}(x, y, z, t)$

	0,001	0,002	0,003	0,004	0,005	0,006	0,007	0,008	0,009
0,001	-7,1E-33	-2,3E-32	-4E-32	-6E-32	-8E-32	-9,5E-32	-1,1E-31	-1,1E-31	-7,8E-32
0,002	-2,2E-32	-7,4E-32	-1,3E-31	-2E-31	-2,7E-31	-3,2E-31	-3,5E-31	-3,5E-31	-2,5E-31
0,003	-3,9E-32	-1,3E-31	-2,4E-31	-3,6E-31	-4,8E-31	-5,7E-31	-6,3E-31	-6,1E-31	-4,2E-31
0,004	-5,6E-32	-1,9E-31	-3,5E-31	-5,3E-31	-7,1E-31	-8,4E-31	-9,1E-31	-8,8E-31	-6,1E-31
0,005	-7,3E-32	-2,5E-31	-4,6E-31	-6,9E-31	-9,2E-31	-1,1E-30	-1,2E-30	-1,1E-30	-7,9E-31
0,006	-8,4E-32	-2,8E-31	-5,3E-31	-8E-31	-1,1E-30	-1,3E-30	-1,4E-30	-1,3E-30	-9,1E-31
0,007	-9,1E-32	-3E-31	-5,6E-31	-8,5E-31	-1,1E-30	-1,3E-30	-1,5E-30	-1,4E-30	-9,8E-31
0,008	-9E-32	-2,9E-31	-5,3E-31	-8E-31	-1,1E-30	-1,3E-30	-1,4E-30	-1,4E-30	-9,8E-31
0,009	-6,4E-32	-2E-31	-3,6E-31	-5,4E-31	-7,2E-31	-8,5E-31	-9,5E-31	-9,8E-31	-7,1E-31



	0,001	0,002	0,003	0,004	0,005	0,006	0,007	0,008	0,009
0,001	1,81E-45	6,65E-45	1,33E-44	2,07E-44	2,74E-44	3,2E-44	3,31E-44	2,91E-44	1,81E-44
0,002	6,6E-45	2,44E-44	4,92E-44	7,63E-44	1,01E-43	1,18E-43	1,22E-43	1,06E-43	6,59E-44
0,003	1,31E-44	4,86E-44	9,82E-44	1,53E-43	2,02E-43	2,36E-43	2,43E-43	2,11E-43	1,3E-43
0,004	2E-44	7,44E-44	1,51E-43	2,35E-43	3,11E-43	3,62E-43	3,72E-43	3,22E-43	1,98E-43
0,005	2,61E-44	9,71E-44	1,97E-43	3,06E-43	4,06E-43	4,73E-43	4,85E-43	4,2E-43	2,58E-43
0,006	3E-44	1,12E-43	2,26E-43	3,52E-43	4,66E-43	5,43E-43	5,58E-43	4,84E-43	2,97E-43
0,007	3,05E-44	1,13E-43	2,29E-43	3,57E-43	4,72E-43	5,51E-43	5,67E-43	4,93E-43	3,03E-43
0,008	2,64E-44	9,77E-44	1,97E-43	3,05E-43	4,04E-43	4,72E-43	4,87E-43	4,26E-43	2,64E-43
0,009	1,63E-44	5,99E-44	1,2E-43	1,86E-43	2,46E-43	2,88E-43	2,98E-43	2,62E-43	1,63E-43

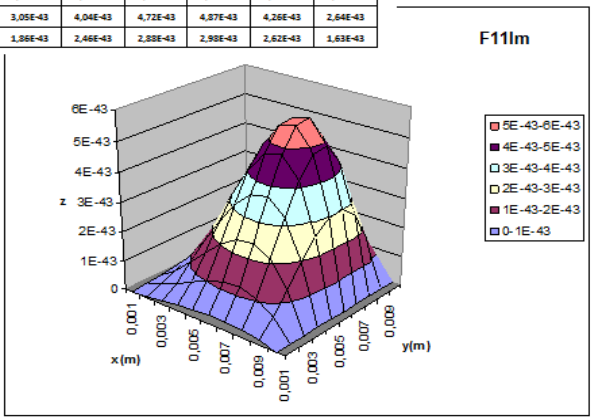
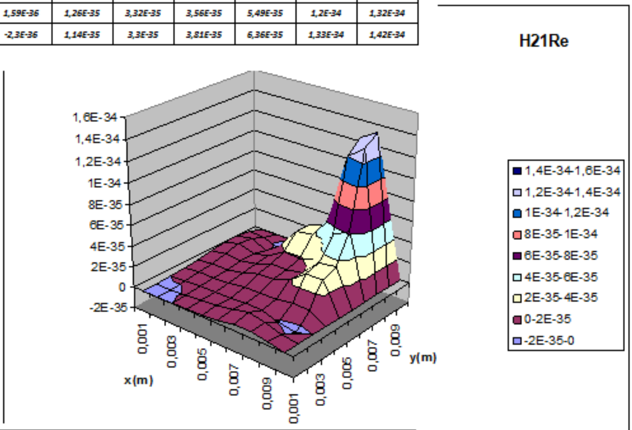


Figure 1. The first scalar component of the vector electric intensity

	0,001	0,002	0,003	0,004	0,005	0,006	0,007	0,008	0,009
0,001	-4,6E-37	-8E-37	-6,8E-38	8,25E-37	1,59E-36	3,35E-36	5,48E-36	5,58E-36	3,25E-36
0,002	-7,9E-37	-1,1E-36	3,82E-37	1,65E-36	2,78E-36	6,18E-36	9,02E-36	6,35E-36	1,37E-36
0,003	-1,3E-38	5,04E-37	1,41E-36	1,68E-36	2,34E-36	4,26E-36	3,82E-36	-1,3E-36	-4,6E-36
0,004	9,8E-37	2,06E-36	2,06E-36	3,28E-36	5,75E-36	6,65E-36	7,1E-36	9,48E-36	8,92E-36
0,005	1,9E-36	3,54E-36	3,24E-36	6,32E-36	1,15E-35	1,31E-35	1,76E-35	3,02E-35	3,05E-35
0,006	3,8E-36	7,33E-36	5,78E-36	7,99E-36	1,4E-35	1,44E-35	1,75E-35	3,31E-35	3,6E-35
0,007	6,1E-36	1,07E-35	6,13E-36	9,45E-36	1,97E-35	1,88E-35	2,34E-35	5,37E-35	6,25E-35
0,008	6,3E-36	8,45E-36	1,59E-36	1,26E-35	3,32E-35	3,56E-35	5,49E-35	1,2E-34	1,32E-34
0,009	3,8E-36	3,03E-36	-2,3E-36	1,14E-35	3,3E-35	3,81E-35	6,36E-35	1,33E-34	1,42E-34



	0,001	0,002	0,003	0,004	0,005	0,006	0,007	0,008	0,009
0,001	1,04E-50	-2,8E-51	-8,8E-50	-2,2E-49	-3,7E-49	-5,4E-49	-6,5E-49	-6E-49	-3,6E-49
0,002	-4,1E-51	-1,1E-49	-3,9E-49	-7,8E-49	-1,2E-48	-1,7E-48	-1,9E-48	-1,7E-48	-9,3E-49
0,003	-9,5E-50	-4,1E-49	-9,3E-49	-1,6E-48	-2,3E-48	-3E-48	-3,2E-48	-2,7E-48	-1,6E-48
0,004	-2,4E-49	-8,4E-49	-1,7E-48	-2,7E-48	-3,8E-48	-4,7E-48	-5,1E-48	-4,8E-48	-3,1E-48
0,005	-4,1E-49	-1,3E-48	-2,5E-48	-3,9E-48	-5,5E-48	-6,8E-48	-7,7E-48	-7,7E-48	-5,3E-48
0,006	-5,9E-49	-1,9E-48	-3,3E-48	-5E-48	-7E-48	-8,8E-48	-1E-47	-1,1E-47	-7,5E-48
0,007	-7,3E-49	-2,2E-48	-3,6E-48	-5,6E-48	-8,1E-48	-1E-47	-1,2E-47	-1,4E-47	-1E-47
0,008	-6,8E-49	-1,9E-48	-3,1E-48	-5,3E-48	-8,1E-48	-1,1E-47	-1,4E-47	-1,7E-47	-1,3E-47
0,009	-4,2E-49	-1,1E-48	-1,9E-48	-3,5E-48	-5,7E-48	-7,8E-48	-1,1E-47	-1,3E-47	-1,1E-47

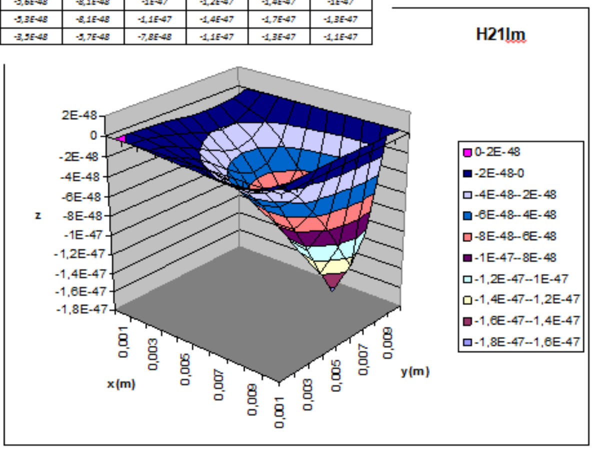


Figure 2. The first scalar component of the vector magnetic intensity

describe the electric and magnetic induction; $\vec{i}(x, y, z, t)$, $\rho(x, y, z, t)$ determine the current (charges) and the charge density; σ, ε, μ – are the positive real constants denoting specific conductivity, electric and magnetic permeability of the medium; $\partial_0 = \partial/\partial t$; $\mathbf{rot}$, div represent the classical differential field operators.

2 Main Results

The method [1] reduces Sys. (1) to the general wave PDE regarding all scalar components of the electromagnetic field vector intensities

$$\left(\Delta + \mu\omega(\omega\varepsilon - i\sigma)\right)\vec{F}_k = -\vec{f}_k, \; (k = 1, 2), \tag{2}$$

where: $\vec{F}_1 = \vec{E}, \vec{F}_2 = \vec{H}$; $\vec{f}_2 = \vec{h}(x, y, z)$, $\vec{f}_1 = -(1/\varepsilon)\,\mathbf{grad}\,\rho$, and $\vec{h}$ is determined by the problem physical viewpoint. The boundary value problem is formulated for Eq. (2) describing electrodynamic process in the finite spatial guided structure. The exact solution of given problem substantially simplifies numerical implementation and computer modeling of the considered electromagnetic phenomenon. The field behavior is shown partially by Figs. 1 and 2.

3 Conclusion

Technique of the present paper allows solving effectively electrodynamic problems with mathematical models in terms of the systems of PDEs.

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Hydromagnetic natural convection flow of fractional nanofluids over a permeable moving heated plate

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Abstract

General exact expressions are established for dimensionless temperature and velocity fields, Nusselt number and the skin friction coefficient corresponding to the hydromagnetic natural convection flow of fractional water based nanofluids past a permeable moving infinite vertical plate with heat generation. The influence of fractional parameter on the heat transfer through the fluid is graphically underlined and discussed when the ramped type heated plate uniformly slides in its plane.

Keywords: natural convection flow, fractional nanofluids.

1 Introduction

It is the well known fact that the fractional models are more flexible in describing the behavior of viscoelastic materials. They can more accurately describe different physical phenomena in comparison to the ordinary models. Our purpose here is to provide exact general solutions for the unsteady hydromagnetic natural convection flow of water based fractional nanofluids past an infinite permeable heated vertical plate with constant heat generation. The obtained results, which are presented under integral form in terms of the Wright function, can be used to generate exact solutions for any flow of this type. Finally, a special case is considered and the influence of fractional parameter on the heat transfer in two fractional nanofluids is graphically underlined and discussed when a ramp type heated plate is uniformly moving in its plane.

2 Formulation of the problem

Let us consider the unsteady natural convection flow of the water based nanofluids past a permeable infinite vertical moving plate with a constant heat source. The nanoparticles of Cu (Copper) or $\mathrm{TiO_2}$ (Titanium oxide) have a uniform shape and size and are in thermal equilibrium state with the base fluid. The plate is in the plane $y=0$ of a fixed Cartesian coordinate system and a uniform magnetic field of strength B is applied along the y-axis. The initial temperature of the whole system is T_∞. At the moment $\mathrm{t}=0^+$ the plate, whose temperature is maintained at the value $T_\infty+\left(T_w-T_\infty\right)g(t),$ begins to slide along the x-axis with a time dependent velocity $Uf(t).$ Here T_w and U are constants while the dimensionless functions $f(\cdot)$ and $\mathrm{g}(\cdot)$ are piecewise continuous and $f(0)=g(0)=0.$ The plate being infinite all physical entities, except the pressure, are functions of y and t only.

Bearing in mind the above assumptions, the dimensionless boundary layer equations of the fractional model in the Boussinesq approximation are given by

$$a_0D_t^\alpha u(y,t)-a_0S\frac{\partial u(y,t)}{\partial y}=\frac{\partial^2 u(y,t)}{\partial y^2}+a_0a_1T(y,t)-a_0a_2u(y,t), \qquad (1)$$

$$b_0D_t^\alpha T(y,t)-b_0S\frac{\partial T(y,t)}{\partial y}=\frac{\partial^2 \mathrm{T}(y,t)}{\partial y^2}-b_0b_1T(y,t);\ \mathrm{y,t}>0, \qquad (2)$$

with the initial and boundary conditions

$$u(y,0)=0,\ T(y,0)=0,\ y\ge 0; u(0,\mathrm{t})=\mathrm{f(t)},\ \mathrm{T}(0,\mathrm{t})=\mathrm{g(t)},\ \mathrm{t}\ge 0, \qquad (3)$$

Here, $u(y,t)$ and $T(y,t)$ are the velocity and the temperature of the fluid, D_t^α is the Caputo fractional derivative and a_0,a_1,a_2,b_0,b_1 are characteristic coefficients of the nanofluid [1].

3 Solution of the problem

The system of fractional partial differential equations (1) and (2), with the initial and boundary conditions (3) is solved using the Laplace transform technique. The corresponding solutions are:

$$T(y,t)=\frac{2}{\sqrt{\pi}}e^{-\frac{b_0S}{2}y}\int_0^\infty \exp\left(-u^2-\frac{b_0b_2y^2}{4u^2}\right)\int_0^t\frac{g(t-s)}{s}W\left(0,-\alpha;-\frac{b_0y^2}{4u^2}s^{-\alpha}\right)dsdu. \quad (4)$$

$$\begin{aligned}u(y,t)=&\frac{y\sqrt{a_0}}{2\sqrt{\pi}}e^{-\frac{a_0S}{2}y}\int_0^\infty\frac{1}{u\sqrt{u}}\exp\left(-\frac{a_0y^2}{4u}-a_3u\right)\int_0^t\frac{f(t-s)}{s}W(0,-\alpha;-us^{-\alpha})dsdu+\\&+\frac{d_0}{2\sqrt{d_1^2-d_2}}\left[\Psi(y,t;b_0,b_2)-\Psi(y,t;a_0,a_3)\right]*\int_0^t\frac{g(t-s)}{s}\int_0^\infty\left[\phi(s;u,p_1)-\phi(s;\mathrm{u},\mathrm{p}_2)\right]e^{-b_2u}duds,\end{aligned} \quad (5)$$

where $W(a,b;\xi)$ is the Wright function [2], the star "*" denotes the convolution product and

$$\phi(t;\mathrm{u},\mathrm{a})=\left[\frac{1}{\sqrt{\pi u}}-ae^{a^2u}erfc\left(a\sqrt{u}\right)\right]W(0,-\alpha;-\mathrm{u}\mathrm{t}^{-\alpha}). \quad (6)$$

$$\Psi(y,t;\mathrm{a},b)=\frac{y\sqrt{a}}{2t\sqrt{\pi}}e^{-\frac{aS}{2}y}\int_0^\infty\frac{1}{u\sqrt{u}}\exp\left(-\frac{ay^2}{4u}-bu\right)W(0,-\alpha;-ut^{-\alpha})du. \quad (7)$$

A simple analysis clearly shows that $T(y,t)$ and $u(y,t)$, given by Eqs. (4) and (5) satisfy all imposed initial and boundary conditions.

The corresponding Nusselt number is given by

$$Nu=\frac{b_0S}{2}g(t)-\frac{\sqrt{b_0}}{2\sqrt{\pi}}\int_0^\infty\frac{e^{-b_2u}}{u\sqrt{u}}\int_0^t\frac{g(t-s)}{s}W(0,-\alpha;-us^{-\alpha})ds. \quad (8)$$

4 Some numerical results and conclusions

In order to get some physical insight of present results, we consider the motion due to a ramp-time heating plate that is moving with uniform velocity. Corresponding solutions are obtained substituting $f(t)$ by $H(t)$ (the Heaviside unit step function) and $g(t)$ by $tH(t)$ in Eqs. (4) and (5). In Figs. 1a and 1b, for comparison temperature profiles corresponding to Cu-water and TiO_2-water fractional/ordinary nanofluids and fractional/ordinary fluids are depicted for different values of fractional parameter α. In both cases the influence of fractional parameter is significant and the heat transfer is stronger for ordinary fluids/nanofluids.

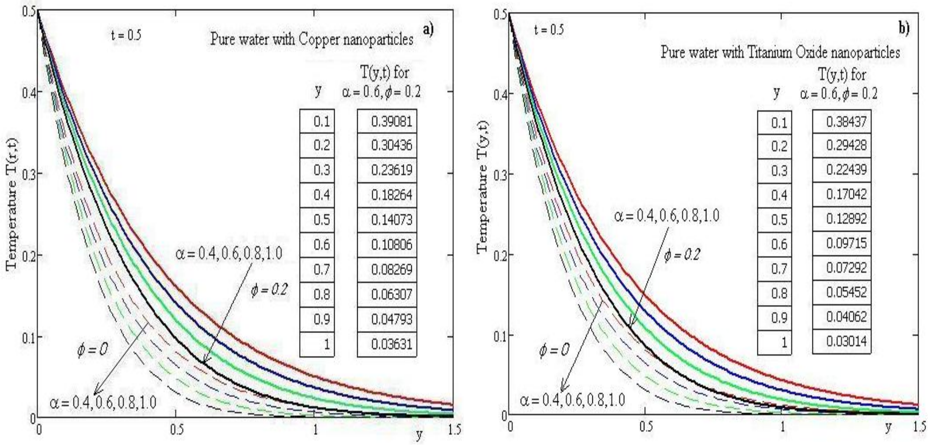

Figure 1. Temperature profiles at different values of fractional parameter

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Parallel algorithm to solve the bimatriceal subgames generated by the informational extended strategies

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Abstract

Parallel algorithm for mixed system with shared and distributed memory to solve bimatrix game generated by the informational extended strategies is described.

Keywords: computer science, game teory, parallel system, parallel algorithm.

According to [1] we can construct the normal form of the bimatrix game

$$\Gamma\left(\theta_1^\alpha, \theta_2^\beta\right) = \left\langle I, J, A^\alpha, B^\beta \right\rangle,$$

where $A^\alpha = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}$, $B^\beta = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}$, $i_j^\alpha \in I^\alpha$, $j_i^\beta \in J^\beta$ to be referred to as informational non-extended game generated by the informational extended strategies $\left(\theta_1^\alpha, \theta_2^\beta\right)$. The game $\Gamma\left(\theta_1^\alpha, \theta_2^\beta\right)$ is played as follows: independently and simultaneously each player $k = \overline{1,2}$ chooses the informational non-extended strategy $i \in I$, $j \in J$, after that the players 1 and 2 calculate the value of the informational extended strategies $i_j^\alpha = \theta_1^\alpha(j)$ and $j_i^\beta = \theta_2^\beta(i)$, after that each player calculates the payoff values $a_{i_j^\alpha j_i^\beta}$, $b_{i_j^\alpha j_i^\beta}$ and with this the game is finished. It is clear that for all strategy profiles (i,j) in the game $\Gamma = \langle A, B\rangle$ the following realization $\left(i_j^\alpha = \theta_1^\alpha(j), j_i^\beta = \theta_2^\beta(i)\right)$ in terms of the informational extended strategies will correspond.

To determine the Nash equilibrium profiles in the bimatrix game of type $\Gamma\left(\theta_1^\alpha, \theta_2^\beta\right)$ we must do the following:

- using the "combinatorial algorithm" construct for all α, β the sets I^α, J^β;
- for all fixed α, β construct the payoff matrices $A^\alpha = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}$, $B^\beta = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}$;
- using existent algorithms determine the set $NE\left(A^\alpha, B^\beta\right)$ of Nash equilibrium profiles in the bimatrix game with the matrices A^α and B^β.

The basic parallel strategy consists of three main steps. The **first step** is to partition the input into several partitions of almost equal sizes. The **second step** is to solve recursively the subproblem defined by each partition of the input. Note that these subproblems can be solved concurrently in the parallel system. The **third step** is to combine or merge the solutions of the different subproblems into a solution for the overall problem. The success of such strategy depends on whether or not we can perform the first and third steps efficiently [2]. To release the first step of the parallel strategy, that is to release data paralelization, we use the open source Scalable Linear Algebra PACKage (ScaLAPACK) [3]. Four basic steps are required to call a ScaLAPACK routine.

- Initialize the process grid.
- Distribute the matrix on the process grid.
- Call ScaLAPACK routine.

We describe the parallel algorithm to determine for all informational extended strategies of the player 1 $\theta_1^\alpha \in \Theta_1 = \{\theta_1^\alpha : J \to I\}_{\alpha=1}^{\varkappa_1}$ and respectively $\theta_2^\beta \in \Theta_2 = \left\{\theta_2^\beta : I \to J\right\}_{\beta=1}^{\varkappa_2}$ of the player 2, the Nash equilibrium profiles of the informational non-extended bimatrix games

$$\begin{aligned}\Gamma\left(\theta_1^{\alpha},\theta_2^{\beta}\right) &= \left\langle I,J,A^{\alpha},B^{\beta}\right\rangle, A^{\alpha}=||a_{i_j^{\alpha}j_i^{\beta}}||_{i\in I}^{j\in J},\\ B^{\beta} &= ||b_{i_j^{\alpha}j_i^{\beta}}||_{i\in I}^{j\in J}, i_j^{\alpha}\in I^{\alpha}, j_i^{\beta}\in J^{\beta}\end{aligned}$$

generated by the informational extended strategies $\left(\theta_1^{\alpha},\theta_2^{\beta}\right)$. For conveniences we introduce the following notations $\widehat{a}_{ij}\equiv a_{i_j^{\alpha}j_i^{\beta}}$ and $\widehat{b}_{ij}\equiv b_{i_j^{\alpha}j_i^{\beta}}$ for all $i\in I$, $j\in J$. So the bimatrix game $\Gamma\left(\theta_1^{\alpha},\theta_2^{\beta}\right)=\left\langle I,J,A^{\alpha},B^{\beta}\right\rangle$ is equivalent to the following bimatrix game $\widehat{\Gamma}\left(\theta_1^{\alpha},\theta_2^{\beta}\right)=\left\langle I,J,\widehat{A}^{\alpha},\widehat{B}^{\beta}\right\rangle$, where $\widehat{A}^{\alpha}=||\widehat{a}_{ij}||_{i\in I}^{j\in J}$, $\widehat{B}^{\beta}=\left\|\widehat{b}_{ij}\right\|_{i\in I}^{j\in J}$. Denote by $NE\left[\widehat{\Gamma}\left(\theta_1^{\alpha},\theta_2^{\beta}\right)\right]$ the set of all Nash strategy profiles in the game $\widehat{\Gamma}\left(\theta_1^{\alpha},\theta_2^{\beta}\right)$.

The parallel algorithm to find the set of all equilibrium profiles

$$((i^*(\alpha,\beta),j^*(\alpha,\beta))\in NE\left[\widehat{\Gamma}\left(\theta_1^{\alpha},\theta_2^{\beta}\right)\right]$$

for all fixed $\alpha=\overline{1,\varkappa_1}$ and $\beta=\overline{1,\varkappa_1}$ consists of the following steps.

1. Using the MPI programming model and open source library ScaLAPACK-BLACS, initialize the processes grid $\{(\alpha,\beta)\}_{\alpha=\overline{1,\varkappa_1}}^{\beta=\overline{1,\varkappa_2}}$, and the root MPI process broadcasts to all $(\alpha,\beta)-$MPI processes the initial matrices $A=||a_{ij}||_{i\in I}^{j\in J}$, and $B=||b_{ij}||_{i\in I}^{j\in J}$ of the bimatrix game $\Gamma=\langle A,B\rangle$.
2. For all fixed type-players α and β all fixed MPI processes (α,β) using the OpenMP directives and combinatorial algorithm construct the sets $\mathcal{I}^{\alpha},\mathcal{J}^{\beta}$ (i.e. construct the informational extended strategies $.\theta_1^{\alpha},\theta_2^{\beta})$ whereupon the matrices $\widehat{A}^{\alpha}=||\widehat{a}_{ij}||_{i\in I}^{j\in J}$, $\widehat{B}^{\beta}=\left\|\widehat{b}_{ij}\right\|_{i\in I}^{j\in J}$, where $\widehat{a}_{ij}=a_{i_j^{\alpha}j_i^{\beta}}$, $\widehat{b}_{ij}=b_{i_j^{\alpha}j_i^{\beta}}$ and $i_j^{\alpha}\in I^{\alpha}, j_i^{\beta}\in J^{\beta}$.
3. All fixed MPI processes (α,β) using the OpenMP functions and ScaLAPACK routines eliminate from matrix $\widehat{A}^{\alpha}$ and $\widehat{B}^{\beta}$ the lines

that are strictly dominated in matrix $\widehat{A}^{\alpha}$ and columns that are strictly dominated in matrix $\widehat{B}^{\beta}$. Finally we obtain the matrices $\left(\widetilde{A}^{\alpha}, \widetilde{B}^{\beta}\right)$, where $\widetilde{A}^{\alpha} = ||\widetilde{a}_{ij}||_{i\in\widetilde{I}}^{j\in\widetilde{J}}$, $\widetilde{B}^{\beta} = ||\widetilde{b}_{ij}||_{i\in\widetilde{I}}^{j\in\widetilde{J}}$ and cardinals $\left|\widetilde{I}\right| \le |I|$, $\left|\widetilde{J}\right| \le |J|$.

4. All fixed MPI processes (α, β) using the OpenMP functions, ScaLAPACK routines and existing algorithm determine all Nash equilibrium strategy profile of the bimatrix game with matrices $\widetilde{A}^{\alpha}$, $\widetilde{B}^{\beta}$ and construct the set $NE\left[\widehat{\Gamma}\left(\theta_1^{\alpha}, \theta_2^{\beta}\right)\right]$ for bimatrix game with matrices $\widehat{A}^{\alpha}$ and $\widehat{B}^{\beta}$.
5. Using ScaLAPACK-BLACS routines, the root MPI process gathers from processes grid $\{(\alpha,\beta)\}_{\alpha=\overline{1,\varkappa_1}}^{\beta=\overline{1,\varkappa_2}}$ the set $NE\left[\widehat{\Gamma}\left(\theta_1^{\alpha}, \theta_2^{\beta}\right)\right]$ of strategy profiles.

For this algorithm a C++ program has been developed using MPI functions, OpenMP directives and ScaLAPACK routines. Program has been testing on the control examples on the Moldova State University HPC cluster. The test results were consistent with theoretical results.

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Modification of the Savage's decision criterion for continuous processes

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Abstract

In decision-making under uncertainty, the decision-maker has no information regarding the occurrence of any states of nature. In this article is presented a modification of the Savage's decision model for the situations when the decision-maker has an infinite number of alternatives and the number of states of nature is finite.

Keywords: decision criteria, function of regrets, states of nature, uncertainty.

1 Introduction

The uncertainty is associated with situations where decisions are taken in conditions with minimum information about the occurrence of uncontrollable factors. These situations occur, as a rule, when the likelihood of uncontrollable factors is unknown and there are no means to determine them. Based on the linear models, the mathematical model and algorithm proposed in this research characterize much of the production activity.

2 Modification of the Savage's decision-making models

In classical game theory, for each pair $(u,\omega)\in U\times\Omega$, a certain utility function $r(u,\omega)$ corresponds to the decision-maker. It is admitted that the set of decision variants U is convex and contains an infinity set of elements and Ω - a finite set of states of nature. Economically speaking, the indicator $r(u,\omega)$ may represent the cost or the income of an economic system.

It is considered a situation, described quantitatively in the form:

$$r(u,\omega)=\sum_{j=1}^{n}C_j(\omega)\cdot u_j \tag{1}$$

$$\begin{cases}\sum_{j=1}^{n}a_{ij}\cdot u_j\le b_i, \quad i=\overline{1,m} & (2)\\ \underline{u_j}\le u_j\le\overline{u_j}, \quad j=\overline{1,n} & (3)\end{cases}$$

where:

$r(u,\omega)$ – the utility expressed in monetary units for the pair (u,ω); *(2) - (3)* – the system of restrictions.

This paper considers the regret criterion or the Savage's criterion [1].

According to Savage [2, 3], the regret is defined as the evaluation of loss by the decider if he does not select the best alternative reported to the realization of a certain state of nature. Therefore, if, for a given state of nature, the guaranteed income is represented by the function (1), which would be obtained for the state $u^*(\omega): r(u^*(\omega),\omega)=\max_u r(u,\omega)$, the new function obtains the following aspect: $r_S(u,\omega)=r(u^*(\omega),\omega)-r(u,\omega)\ge 0$, where $r(u^*(\omega),\omega)-r(u,\omega)$ represents the value of the regret, and $u^*(\omega)$ - the optimal decision for the state ω. In particular circumstances, for some two states of nature the obtained functions have the following aspect:

$$r_S(u,\omega_1)=r(u^*(\omega_1),\omega_1)-r(u,\omega_1)\,, \tag{4}$$

$$r_S(u,\omega_2)=r(u^*(\omega_2),\omega_2)-r(u,\omega_2)\,. \tag{5}$$

The problem of maximizing the objective function (named as function Savage) is:

$$R_S(u)=\max_{\omega\in\Omega}[r_S(u,\omega)]\to\min \tag{6}$$

Firstly, the Simplex method is applied to solve N problems of the following type:

$$R_i(u)=r(u;\omega_i)=\sum_{j=1}^{n}C_j(\omega_i)\cdot u_j\to\max \tag{7}$$

for $i=\overline{1,N}$ with restriction *(2)–(3)*.

It is assumed that $\left(u^i\right)^*$ - optimal solution for problem *i*, $R_i^* = R_i\left(\left(u^i\right)^*\right)$.

It is defined: $r_S\left(u,\omega_i\right) = R_i^* - r\left(u,\omega_i\right)$ - the value of the regret when the decision $u \in U$ is applied, but not the decision $\left(u^i\right)^*$. It can be demonstrated that the functions $r_S\left(u,\omega_i\right)$ and $R_S\left(u\right)$ are convex [4].

Therefore, goal is to solve the problem:

$$R_S\left(u\right) = \max_{1\le i\le N}\left[r_S\left(u,\omega_i\right)\right] \to \min_{u\in U} \tag{8}$$

Applying the method of the generalized gradient [4, 5], there will be described an algorithm to solve the problem of Savage's function on the domain U.

For each $k = 0,1,...,$ there is generated a set of points $u^0, u^1, ..., u^k, u^{k+1}, ... \in U$. Initial point u^0 is given and is taken by the decision-maker from U. Having the approximation of u^k, the next point u^{k+1} is determined as:

$$u^{k+1} = \mathrm{P}_U\left(u^k - h_k \cdot \eta^k\right). \tag{9}$$

Here: $k = 0,1,2,...;$

$$\eta^k = \left\{\begin{array}{l} \mathrm{grad}R_S\left(u^k\right) = -\left(C_1\left(\omega^k\right),...,C_j\left(\omega^k\right),...,C_n\left(\omega^k\right)\right)^T, \\ \qquad\qquad if\ \ \Psi_i(u^k) \le 0\ \forall\, \mathrm{i} = 1,2,...,\mathrm{m}, \\ \qquad\qquad where\ \ \omega^{\mathrm{k}} \in \Omega : r_S\left(u^k,\omega^k\right) = \max\limits_{1\le i\le N} r_S\left(u^k,\omega_i\right) \\ \left(\mathrm{a}_{\mathrm{i_k}1},...,\mathrm{a}_{\mathrm{i_k}j},...,\mathrm{a}_{\mathrm{i_k}n},\right)^T,\ if\ \ \Psi_{i_k}(u^k) > 0. \end{array}\right\}. \tag{10}$$

In order to converge to solution, the series h_k must satisfy the following constraints: $h_k > 0, h_k \to 0, \sum\limits_{\mathrm{k}=0}^{\infty} h_k = \infty.$

If $\Psi_i(u^k) \le 0$, the approximation of u^{k+1} is determined applying the generalized gradient of function $R_S\left(u\right)$, calculated for point $u = u^k$. Otherwise, if $\Psi_{i_k}(u^k) > 0$, the approximation of u^{k+1} is determined applying the generalized gradient of the function Ψ_{i_k}, which is most exceeded in the point $u = u^k$.

3 Conclusion

This article describes the aspects of the importance of decision-making process under uncertainty. Due to this, we propose a modification of classical Savage's criterion for conditions where the decision-maker has an infinite number of alternatives for situations that relate to substantiating and making decisions, described in the terms of linear models. The proposed algorithm can provide effective and real-time solutions to various practical situations when the decision-maker does not have sufficient relevant information on the manifestation of uncontrollable factors.

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Mathematical modeling of the elastic-plastic barrier behavior under high-speed load

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Abstract

The problem of numerical research of the elastoplastic barrier behavior under high-speed loads (arising during projectile explosion) is considered. The results of calculations have shown the mathematical model effectiveness to determine the maximum strength characteristics of the barrier. In addition numerical solutions are useful to identify areas of high stress and predict construction behavior under various loads.

Keywords: mathematical model, elastoplastic deformation, detonation.

1 Introduction

In the present paper we carried out a two-dimensional numerical modeling of elastic-plastic structures behavior under high loads. Mathematical model takes into account the formation and propagation of shock waves, unloading waves, the substance elasticity, plasticity, and other factors.

Physical processes of structural loading are complex and non-stationary. For their modeling and behavior research the systems of partial differential equations and the corresponding models of the of elastic-plastic medium are used.

The possibilities of analytical methods and application of solutions based on physical experiments are quite limited. The progress in the study of complex scientific and technical problems of solid mechanics is dealt with using modern computer technologies and numerical methods. The applied numerical research is optimal for determining high stress zones and minimizing possible risks at various loads.

Researchers are trying to create precise mathematical models, numerical algorithms and data analysis systems to obtain reliable numerical solutions for more efficient designs of resistance constructions. The implementation of these solutions is a complex task because of their large number of parameters.

2 Problem formulation and mathematical model

We consider the non-stationary problem of dynamic loading and the study of the stress-strain state of structures with intensive influence on them, taking into account elastoplastic deformations. The interaction with the physically non-linearly deformable ground surrounding the investigated construction is also taken into account. It is necessary to consider the following factors: the interaction of strain and stress waves, the appearance of plastic deformations and contact interaction of structural elements with the surrounding environment.

The implemented mathematical model is elastic-plastic-damage model [1], [2]. It is well known that under high-loading many materials behave as substances which have both elastic and plastic properties. The computational experiments were carried out according to the finite-difference method, which is the modified Wilkins scheme [3]. The specific feature of the method is the use of the Lagrangian computational grid and a special library of parameters for equations of state for different environments.

3 Results

The modeling results of the shell-free projectile impact on the well walls are described. The projectile is closely located to the well wall.

Consider a ground layer, the plane of which is perpendicular to the axis of the well. The well is located horizontally, and is filled with drilling fluid. Casing walls are made of steel. Outside construction is surrounded by ground.

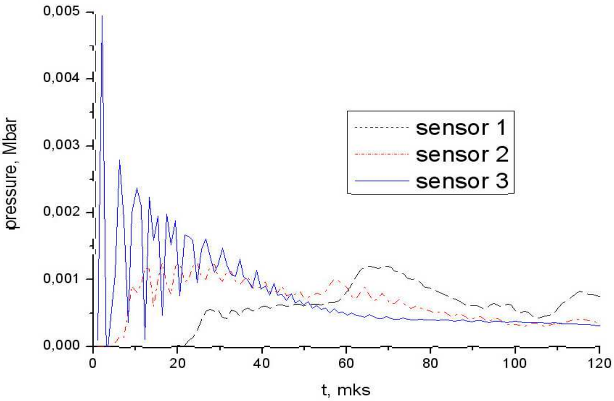

Figure 1. Pressure at control points on the steel casing.

Pressure in three control points on the "steel-ground" boundary are shown in Fig. 1. According to calculations, the maximum pressure is reached at the point closest to the charge. It is 0.0049 MBar for the given initial and boundary conditions.

4 Conclusion

Numerical calculations of the shell-free charge impact on barrier walls have been carried out. The calculations can be useful for optimizing of the explosive charge characteristics in case of gas, oil or water production, by opening productive layers through casing perforation [4].

The modified mathematical model and the developed numerical method allow calculating the stress-strain state of elastoplastic structure under intensive dynamic loads. It makes possible to conduct a wide range of numerical studies in this field.

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Parallel algorithm to find the Bayes-Nash solution in the informational extended game

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Abstract

The Bayes-Nash solutions for informational extended games are discussed. Also the parallel algorithm for mixed system with shared and distributed memory to determine the Bayes-Nash solutions in the bimatrix informational extended games are presented.

Keywords: games, strategies, Bayes-Nash solution, parallel algorithm.

1 Informational extended game

We consider the perfect and complete bimatrix game in strategic form $\Gamma = \langle I, J, A, B \rangle$. According to [1] we can describe the informational extended strategies in bimatrix game as follows. For all fixed $\alpha = 1, ..., n^m$ and $\beta = 1, ..., m^n$ we construct the vectors $\mathbf{i}^\alpha = \left(i_1^\alpha, i_2^\alpha, ..., i_j^\alpha, ..., i_m^\alpha\right)$ and $\mathbf{j}^\beta = \left(j_1^\beta, j_2^\beta, ..., j_i^\beta, ..., j_n^\beta\right)$. The $\mathbf{i}^\alpha$ vector's elements mean the following: if the player 2 will choose the column $j \in J$, then the player 1 will choose the line $i_j^\alpha \in I$. Respectively, the $\mathbf{j}^\beta$ vector's elements mean the following: if the player 1 will choose the line $i \in I$, then the player 2 will choose the column $j_i^\beta \in J$. So we can introduce the following definition. Denote by $I^\alpha = \left\{i_j^\alpha \in I : i_j^\alpha \neq i_k^\alpha, \forall j, k \in J, j \neq k\right\}$ and $J^\beta = \left\{j_i^\beta \in J^\beta : j_i^\beta \neq j_r^\beta \forall i, r \in I, i \neq r\right\}$. Then the set $I^\alpha \subseteq I$, respectively $J^\beta \subseteq J$, is the set of informational non extended strategies of the player 1, respectively 2, generated by the informational extended strategies $\mathbf{i}^\alpha$, respectively $\mathbf{j}^\beta$. Denote by $G\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ the bimatrix game in the informational extended strategies, described above. This game is in the imperfect information on the set of informational extended strategies. According to [2], for bimatrix game $G\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ we construct the Selten-Harsanyi [3] type normal form game $\Gamma^*_{Bayes} = \left\langle K, \{R_k\}_{k\in K}, \{U_k\}_{k\in K} \right\rangle$. Here the set of type-players is $K = K_1 \cup K_2$, where $K_1 = \{\alpha = 1, ..., n^m\}$ and $K_2 = \{\beta = 1, ..., m^n\}$; the sets of pure strategies of the type-players are $R_k = \begin{cases} I^\alpha & k \in K_1, \\ J^\beta & k \in K_2. \end{cases}$; the payoff functions of the type-player k are $U_k = \begin{cases} \mathcal{A}_k\left(\{p(\beta/\alpha)\}_{\beta\in K_2}\right) & k \in K_1, \\ \mathcal{B}_k\left(\{q(\alpha/\beta)\}_{\alpha\in K_1}\right) & k \in K_2, \end{cases}$ where $\mathcal{A}_k\left(\{p(\beta/\alpha)\}_{\beta\in K_2}\right)$ and $\mathcal{B}_k\left(\{q(\alpha/\beta)\}_{\alpha\in K_1}\right)$ are the payoff matrixes of the type-players. In the other words for all fixed "believer probabilities" $p(\beta/\alpha)$ and $q(\alpha/\beta)$, the payoff matrix for the type-players $k \in K_1$ is $\mathcal{A}_k\left(\{p(\beta/\alpha)\}_{\beta\in K_2}\right) = \|\widetilde{a}_{ij}\|_{i\in I}^{j\in J}$, where $\widetilde{a}_{ij} = \sum\limits_{\beta\in K_2} p(\beta/\alpha) a_{i_j^\alpha j_i^\beta}$, and for the type-players $k \in K_2$ is $\mathcal{B}_k\left(\{q(\alpha/\beta)\}_{\alpha\in K_1}\right) = \left\|\widetilde{b}_{ij}\right\|_{i\in I}^{j\in J}$, where $\widetilde{b}_{ij} = \sum\limits_{\alpha\in K_1} q(\alpha/\beta) b_{i_j^\alpha j_i^\beta}$. Here $i_j^\alpha \in I^\alpha$ and $j_i^\beta \in J^\beta$. The Selten-Harsanyi game Γ^*_{Bayes} means the following: for all fixed type-players k_1 and k_2 (i.e. the player 1 chooses the informational extended strategy α and the player 2 chooses the informational extended strategy β) and "believer probabilities" $\{p(\beta/\alpha)\}_{\beta\in K_2}$ of the player 1 (respectively $\{q(\alpha/\beta)\}_{\alpha\in K_1}$ of the player 2) we obtain the bimatrix game $\Gamma^* = \left\langle I, J, \mathcal{A}_{k_1}\left(\{q(\alpha/\beta)\}_{\beta\in K_2}\right), \mathcal{B}_{k_2}\left(\{q(\alpha/\beta)\}_{\alpha\in K_1}\right) \right\rangle$ in the non informational extended strategies. So Selten-Harsanyi game Γ^*_{Bayes} "generates" the big number of the bimatrix games of type Γ^*.

2 Parallel algorithm

According to [2] determining all Bayes-Nash equilibrium profiles, we can determine the all Nash equilibrium profiles for all bimatrix games of type Γ^* in the non extended strategies. So the parallel algorithm to find the all equilibrium profiles consists of the following main steps.

1. Using the MPI programming model, generate the virtual medium of MPI-process communication (MPI Communicator) with linear topology. Root process broadcasts the initial matrices $A = ||a_{ij}||_{i\in I}^{j\in J}$, and $B = ||b_{ij}||_{i\in I}^{j\in J}$ of the bimatrix game $\Gamma = \langle I, J, A, B\rangle$.
2. MPI process with rank k_1 generates the "beliver-probabilities" $p(\beta/\alpha)$ for all β, and MPI process with rank k_2 generates the "beliver-probabilities" $q(\alpha/\beta)$ for all α.
3. Using the MPI programming model and open source library ScaLAPACK-BLACS [4], initialize the processes grid. All fixed MPI processes (α, β) using the OpenMP directives and combinatorial algorithm construct the sets I^α, J^β.
4. MPI process with rank k constructs payoff matrix U_k.
5. Using open source library ScaLAPACK-BLACS, MPI process broadcasts the matrix U_k.
6. All fixed MPI processes, using the OpenMP functions, eliminate from matrix $\mathcal{A}_{k_1}(\cdot)$ and from matrix $\mathcal{B}_{k_2}(\cdot)$ the lines that are strictly dominated in matrix $\mathcal{A}_{k_1}(\cdot)$ and columns that is strictly dominated in matrix $\mathcal{B}_{k_2}(\cdot)$. Finally we obtain the matrices $\widehat{\mathcal{A}}_{k_1}\left(\{p(\beta/\alpha)\}_{\beta\in K_2}\right)$ and $\widehat{\mathcal{B}}_{k_2}\left(\{q(\alpha/\beta)\}_{\alpha\in K_1}\right)$.
7. All fixed MPI processes using the MPI, OpenMP functions, ScaLAPACK routines and existing algorithm, determine all Nash equilibrium profiles in the bimatrix game with matrices $\widehat{\mathcal{A}}_{k_1}(\cdot)$, $\widehat{\mathcal{B}}_{k_2}(\cdot)$ and construct the set of Nash equilibrium profiles in the bimatrix game with matrices $\mathcal{A}_{k_1}(\cdot)$, $\mathcal{B}_{k_2}(\cdot)$.

8. Using ScaLAPACK-BLACS routines, the root MPI process gather the set of Nash equilibrium profiles in the bimatrix game with matrices U_k.

3 Conclusion

For this algorithm a C++ program has been developed using MPI functions, OpenMP directives and ScaLAPACK routines. Program has been tested on the control examples on the Moldova State University HPC cluster. The test results were consistent with theoretical results. In order to determine all sets of Nash equilibrium profiles in bimatrix games generated by information strategies, it is recommended to use exascale HPC systems.

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Conditional Characteristic Functions for the Multidimensional Markov Random Flight

Alexander D. Kolesnik

Abstract

Series representations for the conditional characteristic functions of the symmetric Markov random flight in the Euclidean space $\mathbb{R}^m$, $m \geq 3$, corresponding to one, two and three changes of direction, are given.

Keywords: Markov random flight, conditional characteristic functions, transition density, random walks, higher dimensions.

Consider the stochastic motion of a particle that, at the initial time instant $t = 0$, starts from the origin $\mathbf{0} = (0, \ldots, 0)$ of the Euclidean space $\mathbb{R}^m$, $m \geq 3$, and moves with some constant speed c. The initial direction is a random m-dimensional vector with uniform distribution on the unit sphere $S_1^m = \left\{ \mathbf{x} \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \cdots + x_m^2 = 1 \right\}$.

The motion is controlled by a homogeneous Poisson process $N(t)$ of rate $\lambda > 0$ as follows. At each Poissonian instant, the particle instantaneously takes on a new random direction distributed uniformly on S_1^m and keeps moving with the same speed c until the next Poisson event occurs, then it takes on a new random direction again and so on.

Let $\mathbf{X}(t) = (X_1(t), \ldots, X_m(t))$ be the particle's position at time $t > 0$ which is referred to as the m-dimensional symmetric Markov random flight. At arbitrary time instant $t > 0$ the particle, with probability 1, is located in the closed m-dimensional ball $\mathcal{B}_{ct}^m = \left\{ \mathbf{x} \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \cdots + x_m^2 \leq c^2 t^2 \right\}$.

Consider the distribution function $\Phi(\mathbf{x}, t) = \Pr\left\{ \mathbf{X}(t) \in d\mathbf{x} \right\}, \ \mathbf{x} \in \mathcal{B}_{ct}^m, \ t \geq 0$, of the process $\mathbf{X}(t)$, where $d\mathbf{x} \subset \mathbb{R}^m$ is the infinitesimal element with the Lebesgue measure $\mu(d\mathbf{x}) = dx_1 \dots dx_m$. For arbitrary fixed $t > 0$, the distribution $\Phi(\mathbf{x}, t)$ consists of two components.

The singular component is referred to the case when no Poisson events occur on the time interval $(0, t)$ and it is concentrated on the sphere $S^m_{ct} = \partial \mathcal{B}^m_{ct} = \left\{ \mathbf{x} \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \dots + x_m^2 = c^2t^2 \right\}$. In this case, at time instant t, the particle is located on the sphere S^m_{ct} and the probability of this event is: $\Pr\left\{\mathbf{X}(t) \in S^m_{ct}\right\} = e^{-\lambda t}$.

If at least one Poisson event occurs on the time interval $(0, t)$, then the particle is located strictly inside the ball $\mathcal{B}^m_{ct}$ and the probability of this event is: $\Pr\left\{\mathbf{X}(t) \in \text{int } \mathcal{B}^m_{ct}\right\} = 1 - e^{-\lambda t}$.

The part of $\Phi(\mathbf{x}, t)$ corresponding to this case is concentrated in the interior: $\text{int } \mathcal{B}^m_{ct} = \left\{ \mathbf{x} \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \dots + x_m^2 < c^2t^2 \right\}$ of the ball $\mathcal{B}^m_{ct}$ and forms its absolutely continuous component.

Let $p(\mathbf{x}, t), \ \mathbf{x} \in \mathcal{B}^m_{ct}, \ t > 0$, be the density of distribution $\Phi(\mathbf{x}, t)$. It has the form: $p(\mathbf{x}, t) = p^{(s)}(\mathbf{x}, t) + p^{(ac)}(\mathbf{x}, t), \ \mathbf{x} \in \mathcal{B}^m_{ct}, \ t > 0$, where $p^{(s)}(\mathbf{x}, t)$ is the density of the singular component of $\Phi(\mathbf{x}, t)$ concentrated on S^m_{ct} and $p^{(ac)}(\mathbf{x}, t)$ is the density of the absolutely continuous component of $\Phi(\mathbf{x}, t)$ concentrated in $\text{int } \mathcal{B}^m_{ct}$.

The singular part of density is given by the formula: $p^{(s)}(\mathbf{x}, t) = \frac{e^{-\lambda t}\, \Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}(ct)^{m-1}} \delta(c^2t^2 - \|\mathbf{x}\|^2), \ \mathbf{x} \in \mathbb{R}^m, \ t > 0$, where $\delta(x)$ is the Dirac delta-function. This is the density of the uniform distribution on S^m_{ct}.

The absolutely continuous part of density has the form: $p^{(ac)}(\mathbf{x}, t) = f^{(ac)}(\mathbf{x}, t)\Theta(ct - \|\mathbf{x}\|), \ \mathbf{x} \in \mathbb{R}^m, \ t > 0$, where $f^{(ac)}(\mathbf{x}, t)$ is some positive function absolutely continuous in $\text{int } \mathcal{B}^m_{ct}$ and $\Theta(x)$ is the Heaviside unit-step function.

Various properties of the Markov random flight $\mathbf{X}(t)$ were studied in [1, 2, 3]. However, the explicit formulas for the density of $\mathbf{X}(t)$ were obtained in a few dimensions only. In the following theorem we present series representations for conditional characteristic functions of $\mathbf{X}(t)$.

Theorem. *For arbitrary dimension $m \geq 3$, the conditional characteristic functions $H_1(\boldsymbol{\alpha}, t), \ H_2(\boldsymbol{\alpha}, t), \ H_3(\boldsymbol{\alpha}, t)$ of the symmetric Markov random flight $\mathbf{X}(t)$ corresponding to one, two and three changes of directions, respectively, are given by the formulas:*

$$H_1(\boldsymbol{\alpha}, t) = \sqrt{\frac{\pi}{2}} \, \frac{(m-2)\,\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} \times \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{m-1}{2}\right)\,(ct\|\boldsymbol{\alpha}\|)^{k-1/2}}{(2k)!!\,\Gamma\left(k+\frac{m}{2}\right)\,(k+m-2)} \, J_{k+1/2}(ct\|\boldsymbol{\alpha}\|), \tag{1}$$

$$H_2(\boldsymbol{\alpha}, t) = \frac{3(m-2)^2\,\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} \times \sum_{k=0}^{\infty} \frac{\xi_k\,(ct\|\boldsymbol{\alpha}\|)^{k-1}}{2^k\,(2k+3(m-2))\,\Gamma\left(k+\frac{3}{2}\right)} \, J_{k+1}(ct\|\boldsymbol{\alpha}\|), \tag{2}$$

$$H_3(\boldsymbol{\alpha}, t) = 12\sqrt{\frac{\pi}{2}} \left(\frac{(m-2)\,\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)}\right)^2 \times \sum_{k=0}^{\infty} \frac{\eta_k\,(ct\|\boldsymbol{\alpha}\|)^{k-3/2}}{(2k+2)!!\,(k+2(m-2))} \, J_{k+3/2}(ct\|\boldsymbol{\alpha}\|), \tag{3}$$

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m, \quad \|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \dots + \alpha_m^2}, \quad m \geq 3, \quad t > 0,$$

where $J_\nu(z)$ are the Bessel functions and the coefficients ξ_n, η_n are given by the formulas:

$$\xi_k = \sum_{l=0}^{k} \frac{\Gamma\left(k-l+\frac{1}{2}\right)\,\Gamma\left(l+\frac{m-1}{2}\right)}{(k-l)!\,\Gamma\left(l+\frac{m}{2}\right)\,(l+m-2)}, \qquad k \geq 0,$$

$$\eta_k = \sum_{l=0}^{k} \frac{\Gamma\left(k-l+\frac{m-1}{2}\right)\,\Gamma\left(l+\frac{m-1}{2}\right)}{\Gamma\left(k-l+\frac{m}{2}\right)\,\Gamma\left(l+\frac{m}{2}\right)\,(l+m-2)}, \qquad k \geq 0.$$

Corollary. *For arbitrary dimension $m \geq 3$, the inverse Fourier transformation of the conditional characteristic function $H_1(\boldsymbol{\alpha}, t)$ yields the conditional density $p_1(\mathbf{x}, t)$ corresponding to the single change of direction that has the form:*

$$
\begin{aligned}
p_1(\mathbf{x},t) = \mathcal{F}_{\boldsymbol{\alpha}}^{-1} \Bigg[\sqrt{\frac{\pi}{2}} \, \frac{(m-2)\,\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)} \\
\times \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{m-1}{2}\right)\,(ct\|\boldsymbol{\alpha}\|)^{k-1/2}}{(2k)!!\,\Gamma\left(k+\frac{m}{2}\right)\,(k+m-2)} \, J_{k+1/2}(ct\|\boldsymbol{\alpha}\|) \Bigg](\mathbf{x}) \\
= \frac{2^{m-3}\,\Gamma(\frac{m}{2})}{\pi^{m/2}\,(ct)^m} \, F\left(\frac{m-1}{2}, -\frac{m}{2}+2; \frac{m}{2}; \frac{\|\mathbf{x}\|^2}{c^2t^2}\right) \Theta(ct-\|\mathbf{x}\|), \qquad (4) \\
\mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m, \qquad m \geq 3, \qquad t > 0,
\end{aligned}
$$

where $F(\alpha,\beta;\ \gamma;\ z)$ is the Gauss hypergeometric function and $\Theta(x)$ is the Heaviside unit-step function.

One can check that the series in formulas (1), (2) and (3) are convergent for any fixed $t > 0$. However, inverting functions (2) and (3) in $\boldsymbol{\alpha}$ is a very difficult problem.

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Monte Carlo Simulation of the Finite-Velocity Random Walks in One and Two Dimensions

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Abstract

Computer simulation of the finite-velocity random motions on the line $\mathbb{R}^1$ and in the plane $\mathbb{R}^2$ based on Monte Carlo algorithms, is done. The main statistical characteristics of the processes obtained by using such algorithms, are presented.

Keywords: Finite-velocity random walks, computer simulation, Monte Carlo algorithms, statistical estimates.

1. Preliminaries. It is known (see [1, formula (3.7)]) that the probability distribution function (PDF) of the Goldstein-Kac telegraph process $X(t)$, represented by a stochastic motion with constant speed $c > 0$ on the real line $\mathbb{R}^1$ and driven by a homogeneous Poisson process of rate $\lambda > 0$, for arbitrary $x \in (-ct, ct]$, $t > 0$, is given by the formula:

$$\begin{aligned} &\Pr\{X(t) < x\} \\ &= \frac{1}{2} + \frac{\lambda x e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k}}{2^{2k}(k!)^2} \left(1 + \frac{\lambda t}{2k+2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{c^2 t^2}\right), \end{aligned} \tag{1}$$

where $F(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function.

The absolutely continuous part of the PDF of the planar Markov random flight $\mathbf{X}(t) = (X_1(t), X_2(t))$ has the form (see [2, formula (20)]):

$$\Pr\{\mathbf{X}(t) \in d\mathbf{x}\} = \frac{\lambda}{2\pi c} \frac{\exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - \|\mathbf{x}\|^2}\right)}{\sqrt{c^2t^2 - \|\mathbf{x}\|^2}} \mu(d\mathbf{x}), \tag{2}$$

$$\mathbf{x} = (x_1, x_2) \in \operatorname{int} \mathbf{C}(\mathbf{0}, ct), \; \mu(d\mathbf{x}) = dx_1 dx_2, \; \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}, \; t > 0,$$

where $\mathbf{C}(\mathbf{0}, ct) \subset \mathbb{R}^2$ is the disc of radius ct centred at the origin $\mathbf{0} \in \mathbb{R}^2$.

Using (2), the following formula was obtained (see [2, Remark 2]):

$$\Pr\{\mathbf{X}(t) \in \mathbf{C}(\mathbf{0}, r)\} = 1 - \exp\left(-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - r^2}\right), \qquad 0 \le r < ct, \tag{3}$$

yielding the probability of being, at time instant $t > 0$, in arbitrary disc $\mathbf{C}(\mathbf{0}, r)$ of radius $r < ct$ centred at the origin $\mathbf{0}$.

While the PDFs (1) and (2) are of a great interest, their application for evaluating many important characteristics, such as the probabilities of being in some curvilinear subsets (for example, in a subdisc $\mathbf{C}(\mathbf{x}^0, r) \subset \mathbf{C}(\mathbf{0}, ct)$ with shifted centre $\mathbf{x}^0 \neq \mathbf{0}$), is a fairly difficult and sometimes impracticable analytical and computational problem. To overcome this difficulty, the Monte Carlo algorithms were built for simulating the telegraph process $X(t)$ and the symmetric planar Markov random flight $\mathbf{X}(t)$. Based on these algorithms, a computer software was created that enables us to obtain, with very good accuracy, the statistical estimates of some their important characteristics, such as the distributions, expectations, variances, the probabilities of being in some subsets, etc. The language C^{++} was used for modelling the processes and calculating the statistical estimates of their basic characteristics. Some results of this simulation are presented below.

2. Simulation of the telegraph random process. A computer program for simulating the Goldstein-Kac telegraph process $X(t)$ on the real line with arbitrary constant speed $c > 0$ and arbitrary intensity of switchings $\lambda > 0$, was elaborated. The results of the simulation related to the PDF of $X(t)$ are given in Table 1 below.

Statistical estimates of PDF (third column) obtained for various values of spatial variable x are compared with the respective exact values of PDF (second column) calculated by means of analytical formula (1). We see that our simulation program yields very good accuracy with stabilization at the fourth digit. To reach this accuracy, $3 \cdot 10^8$ independent realizations of $X(t)$ were generated. Calculation of each statistical estimate in Table 1 takes about 2 minutes.

Values of x	Exact values of PDF	Statistical estimates of PDF
-3.0	0.14777	0.14780
-2.5	0.19661	0.19660
-2.0	0.25059	0.25060
-1.5	0.30892	0.30892
-1.0	0.37065	0.37063
-0.5	0.43473	0.43473
0.0	0.50000	0.49999
0.5	0.56527	0.56526
1.0	0.62935	0.62939
1.5	0.69108	0.69106
2.0	0.74941	0.74944
2.5	0.80339	0.80334
3.0	0.85223	0.85227

Table 1. *PDF of the telegraph process* $X(t)$ *(for* $c = 2, \lambda = 1, t = 2$*)*

3. Simulation of the planar random flight. A computer program was created for simulating the planar Markov random flight $\mathbf{X}(t)$ with uniform choice of directions whose absolutely continuous part of PDF is given by (2) (see [2, Theorem 2]). Statistical estimates for the probabilities of being in the disc $\mathbf{C}(\mathbf{0}, r)$ of radius $r < ct$ centred at the

Values of radius r	Exact probability	Statistical estimates
1	0.01038	0.01038
2	0.04108	0.04107
3	0.09085	0.09085
4	0.15767	0.15765
5	0.23877	0.23877
6	0.33097	0.33100
7	0.43067	0.43065
8	0.53417	0.53416
9	0.63785	0.63789
10	0.73860	0.73860
11	0.83487	0.83490

Table 2. *Probabilities* $Pr\{\mathbf{X}(t) \in \mathbf{C}(\mathbf{0}, r)\}$ *(for* $c = 4, \lambda = 1, t = 3$*)*

origin $\mathbf{0}$ and their comparison with exact probabilities calculated by means of analytical formula (3), are given in Table 2. In Table 3 the statistical estimates for the probabilities of being in the disc $\mathbf{C}(\mathbf{x}^0, r)$ with different shifted centres $\mathbf{x}^0 \neq \mathbf{0}$ and radii, are given. Note that no explicit analytical formulas exist for such cases of shifted centres.

Centre $\mathbf{x}^0 = (x_1^0, x_2^0)$	Values of radius r	Statistical estimates
$(1, 1)$	2	0.04052
$(-1, 3)$	4	0.14730
$(-1, 9)$	2.5	0.03523
$(7, -4)$	3	0.05713
$(0, 11)$	1	0.00454
$(5, -7)$	0.5	0.00151
$(3, -4)$	7	0.36964
$(9, -7)$	0.5	0.00109
$(0, 5)$	3	0.07635
$(1, 4)$	4	0.14036

Table 3. *Probabilities* $Pr\{\mathbf{X}(t) \in \mathbf{C}(\mathbf{x}^0, r)\}$ *(for* $c = 4, \lambda = 1, t = 3$*)*

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Stationary Nash Equilibria for Stochastic Games

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Abstract

The problem of the existence of stationary Nash equilibria in stochastic games with finite state and action spaces is considered. Necessary and sufficient conditions for the existence of stationary Nash equilibria in the considered class of games are presented.

Keywords: Stochastic games, stationary strategies, Nash equilibrium, optimal stationary strategies.

1 Introduction

In this paper we consider stochastic games with average and discounted payoffs. A stochastic game consists of the following elements [1, 2]:

- a state space X (which we assume to be finite);
- a finite set $A^i(x)$ of actions with respect to each player $i \in \{1, 2, \ldots, n\}$ for an arbitrary state $x \in X$;
- a payoff $f^i(x, a)$ with respect to each player $i \in \{1, 2, \ldots, n\}$ for each state $x \in X$ and for an arbitrary action vector $a \in \prod_i A^i(x)$;
- a transition probability function $p : X \times \prod_{x\in X} \prod_{i=1}^{n} A^i(x) \times X \rightarrow [0, 1]$ that gives the probability transitions $p_{x,y}^a$ from an arbitrary $x \in X$ to an arbitrary $y \in Y$ for a fixed action vector $a \in \prod_i A^i(x)$, where $\sum_{y\in X} p_{x,y}^a = 1, \quad \forall x \in X, \ a \in \prod_i A^i(x)$;
- a starting state $x_0 \in X$.

The game starts in the state x_0 and the play proceeds in *a sequence* *of stages.* At *stage* t players observe state x_t and simultaneously and independently choose actions $a_t^i \in A^i(x_t), \ \ i = 1, 2, \dots, n$. Then nature selects state $y = x_{t+1}$ according to probability transitions $p_{x_t,y}^{a_t}$ for given action vector $a_t = (a_t^1, a_t^2, \dots, a_t^n)$. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \dots, x_t, a_t, \dots$ that defines the corresponding stream of stage payoffs $f_t^1 = f^1(x_t, a_t),\ f_t^2 = f^2(x_t, a_t), \dots,\ f_t^n(x_t, a_t) \quad t = 0, 1, 2, \dots,$ where f_t for $t \geq 1$ are random variables with probability distributions in the state-stage induced by the stochastic process with given starting state x_0 and actions $a_t,\ t = 0, 1, 2, \dots$. *The average stochastic game* is the game with payoffs of players

$$\omega_{x_0}^i = \lim_{t\to\infty} \inf \ \mathsf{E}\left(\frac{1}{t}\sum_{\tau=0}^{t-1} f_\tau^i\right), \quad i = 1, 2, \dots, n,$$

The stochastic game with discounted sum of stage payoffs is the game with payoffs of players

$$\sigma^i = \lim_{t\to\infty} \inf \ \mathsf{E}\left(\sum_{\tau=1}^{t-1} \lambda^\tau f_\tau^i\right), \quad i = 1, 2, \dots, n,$$

where $\lambda,\ 0 < \lambda < 1,$ is a given discount factor.

2 Stochastic games in stationary strategies

A strategy of player $i \in \{1, 2, \dots, n\}$ in a stochastic game is a mapping s^i that for every state $x_t \in X$ provides a probability distribution over the set of actions $A^i(x_t)$. If these probabilities take only values 0 and 1, then s^i is called *pure strategy*, otherwise s^i is called *mixed strategy.* If these probabilities depend only on the state $x_t = x \in X$ (i. e. s^i do not depend on t), then s^i is called *stationary strategy.* This means the set of stationary strategies S^i of player i can be regarded as the set of solutions of the following system

$$\begin{cases} \displaystyle\sum_{a^i \in A^i(x)} s_{x,a^i}^i = 1, & \forall x \in X; \\ \qquad\quad s_{x,a^i}^i \geq 0, & \forall x \in X, \ \ \forall a^i \in A^i(x). \end{cases}$$

The stochastic game with average payoffs in the terms of stationary strategies is formulated as follows: Let $\overline{S}^i$, $i \in \{1, 2, \dots n\}$ be the corresponding stationary strategies of the players $1, 2, \dots, n$. On $\overline{S} = \overline{S}^1 \times \overline{S}^2 \times \dots \times \overline{S}^n$ we define n payoff functions

$$\psi^i_\theta(s^1, s^2, \dots, s^n) = \sum_{x \in X} \sum_{(a^1, a^2, \dots, a^n) \in A(x)} \prod_{k=1}^{n} s^k_{x,a^k} f^i(x, a^1, a^2 \dots a^n) q_x, \quad i = 1, 2, \dots, n$$

where $q_x (x \in X)$ are determined uniquely from the following system of linear equations

$$\begin{cases} q_y - \sum\limits_{x \in X} \sum\limits_{(a^1, a^2, \dots, a^n) \in A(x)} \prod\limits_{k=1}^{n} s^k_{x,a^k} p^{(a^1, a^2, \dots, a^n)}_{x,y} q_x = 0, & \forall\, y \in X; \\ q_y + w_y - \sum\limits_{x \in X} \sum\limits_{(a^1, a^2, \dots, a^n) \in A(x)} \prod\limits_{k=1}^{n} s^k_{x,a^k} p^{(a^1, a^2, \dots, a^n)}_{x,y} w_x = \theta_y, & \forall y \in X \end{cases}$$

for an arbitrary fixed profile $s = (s^1,\ s^2, \dots,\ s^m) \in \overline{S}$. The functions $\psi^i_\theta(s^1, s^2, \dots, s^n)$, $i = \overline{1, n}$ represent the payoff functions for the average stochastic game in normal form $\langle \{\overline{S}^i\}_{i=\overline{1,n}}, \{\psi^i_\theta(s)\}_{i=\overline{1,n}} \rangle$. Here $\theta_y \geq 0$ for $y \in X$ and $\sum_{y \in X} \theta_y = 1$. If $\theta_y = 0,\ \forall y \in X \setminus \{x_0\}$ and $\theta_{x_0} = 1$, then we obtain an average stochastic game with starting state x_0. If $\theta_y > 0,\ \forall y \in X$ and $\sum_{y \in X} \theta_y = 1$, then we obtain an average stochastic game when the play starts in the states $y \in X$ with probabilities θ_y. *The stochastic game with discounted payoffs in the terms of stationary strategies* is formulated as follows: Let $\overline{S}^i$, $i \in \{1, 2, \dots n\}$ be the corresponding stationary strategies of the players $1, 2, \dots, n$. On $\overline{S} = \overline{S}^1 \times \overline{S}^2 \times \dots \times \overline{S}^n$ we define n payoff functions

$$\varphi^i(s^1, s^2, \dots, s^m) = \sum_{y \in Y} \theta_y \sigma^i_y, \qquad i = 1, 2, \dots, n$$

where σ^i_x for $x \in X$ satisfy the conditions

$$\sigma^i_x - \lambda \sum_{y \in X} \sum_{(a^1, a^2, \dots, a^k) \in A(x)} \prod_{k=1}^{n} s^k_{x,a^k}\, p^{(a^1, a^2, \dots, a^n)}_{x,y} \sigma^i_y = \sum_{(a^1, a^2, \dots, a^k) \in A(x)} \prod_{k=1}^{n} s^k_{x,a^k} f^i(x, a^1, a^2 \dots, a^n), \quad \forall x \in X, i = \overline{1, n}.$$

The functions $\sigma^1_{x_0}(s), \sigma^2_{x_0}(s), \dots, \sigma^n_{x_0}(s)$ on $\overline{S} = \overline{S}^1 \times \overline{S}^2 \times \cdots \times \overline{S}^n$ define a game in normal form $\langle \{\overline{S}^i\}_{i=\overline{1,n}}, \{\sigma^i_{x_0}(s)\}_{i=\overline{1,n}} \rangle$.

Using the above models we derived necessary and sufficient conditions for the existence of stationary Nash equilibria in the case of two-player average stochastic games and in the case of n-player stochastic games with discounted payoffs.

3 Conclusion

For an arbitrary average stochastic game with two players and for an arbitrary n-player stochastic game with discounted payoffs stationary Nash equilibria exist. The presented game models in normal form can be used for determining stationary Nash equilibria.

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On Stability Improvement of Environment's Dynamic Systems

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Abstract

The stability improvement of environment's dynamic system evolution is described on general case of dynamic systems that depend on parameters. The important property of stable regions separation from the free parameters domain of the dynamic system, accepted for all dynamic systems from the literature without mathematical justification is detailed here calling theoretical results from some branches of mathematics.

Some mathematical conditions on separation are identified using the study on matrix components functions that define the linear dynamic system because some properties of such functions can be transmitted to matrix eigenvalues. One case of specified dynamic system is described.

Keywords: environment, dynamic system, stable region separation, mathematical model.

1 Introduction

The aim of the paper is to emphasize the mathematical characterization aspects of the environment through the mathematical characterization of the dynamic systems that approach the phenomena from the reality. The mathematical branches that offer the resources of characterization can be from real analysis theory, dynamic systems theory, matrix theory, etc.

In each dynamic system that depends on parameters from the literature it is exposed the property of separation between stable and unstable regions in the free parameters domain of the dynamic system. We observe for a dynamic system model which approaches the phenomenon from the environment that has the mathematical property of separation between stable and unstable region in the free parameters domain of the system, property confirmed by the dynamic systems analyzed in the literature.

The stable regions existence in the free parameters domain is an important property of the environment's dynamic system that permits selection of compatible criterion on the parameters in the stable region for stable evolution optimization, in other words permits stability control.

2 On environment's dynamic system model

The linear dynamic system or the "first approximation" of the nonlinear dynamic system, in the general case as depending on parameters, is described by real matrix for the examples from the literature [1]-[4]. The components of such real matrix are assumed to be piecewise continuous functions of the system parameters.

The properties on the stability for the dynamic system that depend on parameters are defined by the properties of eigenvalues functions for the system matrix. The eigenvalues of the system matrix are the same as for the matrix in Hessenberg corresponding form. The Hessenberg form of the matrix is recognized by the condition $a_{ij} = 0$ for $2 < i \le n$, $j < i - 1$. The dynamic system can be substituted by the dynamic system using equivalent Hessenberg form of the system matrix that facilitates the stability analysis of the system.

The system matrix in Hessenberg form is denoted by A. The matrix $A - \lambda I$, where λ is real or complex value and I unity matrix, is also a matrix in Hessenberg form. The value λ defines "the shift of origin" for the matrix. The shift of origin for the matrix is important because it allows the transposition of the real matrix that defines the dynamic system in the complex domain through the complex value λ.

The QR algorithm for the matrix A with the shift of origin is defined by the relations:

$$Q_s(A_s - k_s I) = R_s,\ A_{s+1} = R_s Q_s^T + k_s I = Q_s A_s Q_s^T,\ s = 1, 2, \ldots \qquad (1)$$

In the above relations, by A_1 the initial matrix A of the system in Hessenberg form is denoted, k_s is "shift of origin", Q_s is orthogonal matrix, R_s is upper triangular matrix, A_s, $s \ge 2$ is also in Hessenberg form.

The shift of origin, with the initial value λ sufficiently close to one initial matrix eigenvalue, real or complex, imposes acceleration of the QR algorithm convergence to respective eigenvalue on the similar diagonal form of the matrix. This is another important motivation for using QR algorithm with the shift of origin.

Some aspects on the stability study for the dynamic system defined by autonomous equation $\mathrm{d}\,x/\mathrm{d}\,t = f(x)$ that accepts $x = 0$ as solution. Many particular dynamic systems are of the autonomous form. The function $f(x) = (f_1(x), ..., f_n(x))^T$ is assumed to depend on n dimensional variable x and admit the Taylor expansion near the origin so that:

$$\mathrm{d}\,x/\mathrm{d}\,t = \left[a_{ij}\right]x + g(x);\; a_{ij} = \partial f_i(x)/\partial x_j\Big|_{x=0};\; i, j = 1, ..., n. \qquad (2)$$

The following theorem is due to Liapunov:

Theorem 1 [1], [2]. *The evolution of the dynamic system (2) is asymptotic stable in origin if the real parts of all eigenvalues of the matrix* $A = \left[a_{ij}\right]$, $i, j = 1, ..., n$ *are strictly negative. The evolution of the dynamic system (2) is unstable in origin if the real part of at least one eigenvalue of the matrix* $A = \left[a_{ij}\right]$, $i, j = 1, ..., n$ *is strictly positive.*

The above theorem analyzes only punctual stability of the system.

Below we formulate the theorem on separation between stable and unstable regions from the free parameters domain of the dynamic system.

Theorem 2 [3]. *If the linear dynamic system defined by the real matrix* A*, in the Hessenberg form, has the piecewise continuous components of the matrix as functions of the dynamic system free parameters and the QR algorithm with the shift of origin in complex domain is convergent to the similar diagonal form corresponding to the matrix* A *and assures that the real part of the eigenvalue functions from the diagonal form are also piecewise continuous, then these conditions impose the separation between stable and unstable regions of the dynamic system in the free parameters domain.*

3 On particular dynamic system model

We describe the evolution imposed to walking biped robot by kinematics evolution proposed by Cononovici (2016) to each leg, here in vertical plane, defined by the pivot point Bt, knee joint Qt and base point Pt.

The base point Pt is moving on the selected ellipse arc between points $\mathrm{P_I}$ and $\mathrm{P_F}$, in cycling evolution of biped walking robot, using uniform accelerated displacement on the horizontal direction up to the median point $\mathrm{P_M}$, for one leg, and symmetric displacement assured up to the final point $\mathrm{P_F}$ of the robot leg. The joint point Bt attached to the body of the robot is moving simultaneously with point Pt, having linear route parallel to the axis Ox, using uniform displacement at each cycle.

The evolution imposed for knee joint Qt, can be also improved by selection of parameters from the free parameters domain of the system to respect one compatible criterion of optimization.

4 Conclusion

The general case of the dynamic systems that depend on parameters from the environment is analyzed. The separation of stable regions from the free parameters domain is a fundamental mathematical property of the systems that can be accepted as first axiom of the environment. The possibility of stable evolution improvement is emphasized.

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Numerical modeling of the performance characteristics for exhaustive polling models

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Abstract

Some analytical results for exhaustive polling models with DD priority, such as distribution of busy period and auxiliary characteristics are presented. Numerical solutions for k - busy period are obtained and a numerical example is presented.

Keywords: polling system, DD priority, Laplace-Stieltjes transform, busy period.

1 Introduction

It is well known that polling models find different applications in various fields, such as telecommunications, economy, industry, etc. Priority queueing systems are a large class of queueing systems where the requests that enter into the system are distinguished by their importance.

Let's consider a queueing system $M_r|G_r|1|\infty$ with DD (Discretionary Discipline) priority: if the service time of a_k–request is less than set value $\theta_k, (k = 2, \ldots, r)$, then the arrived request with priority higher k (σ_{k-1}–request) achieves absolute priority, otherwise – the relative one. The durations of service a_k–requests are independent random variables B_k with distribution function $B_k(x)$, $(k = 1, \ldots, r)$.

The switching takes place only at service's interruption and at returning to the interrupted service. If the service of a_j-request is interrupted by arriving a_i-request, $i < j$, then, at once switching to L_i $(\to i)$ flow begins. When the system will be free from requests of priority higher than j, the switching $(\to j)$ begins, and only then the server is ready to serve the interrupted request. The durations of switching ($\to i$) are random variables C_i with distribution function $C_i(x)$, $(i = 1, \dots, r)$. The variables B_k and C_i are independent. An arbitrary switching ($\to k$) also may be interrupted by arriving σ_{k-1}-request.

Remark 1. *For models with 2 priorities classes and non-zero switching time the DD discipline was considered in [1].*

Remark 2. *The interruptions within service and switching processes generate a large class of models (schemes) which we will note by indexes I.J, where I reflects the evolution of interrupted service, but J – the evolution of interrupted switching [2].*

2 Busy period distribution

We'll introduce notations. We'll denote by $\Pi(x)$, $\overline{\Pi}_k(x)$, $\overline{\Pi}_{kk}(x)$, $H_k(x)$, $\overline{\Pi}_{kk}^{(n)}(x)$, $N_k(x)$, $\Pi_k(x)$, $\Pi_{kk}(x)$ the distribution function of busy period, k–period, kk–period, k–service cycle, kkn–period, k–cycle of switching, Π_k–period, Π_{kk}–period and by $\pi(s)\dots\pi_{kk}(s)$ – their Laplace-Stieltjes transforms (the definition of these see [3]). Let's consider also $\sigma_k = a_1 + \dots + a_k$, where a_k–the parameter of Poisson flow of k–th priority.

Theorem 1. *For all schemes*

$$\sigma_k \overline{\pi}_k(s) = \sigma_{k-1}\overline{\pi}_{k-1}(s + a_k - a_k\overline{\pi}_{kk}(s)) + a_k\pi_{kk}(s), \tag{1}$$

$$\overline{\pi}_{kk}(s) = h_k(s + a_k - a_k\overline{\pi}_{kk}(s)), \tag{2}$$

$$\begin{aligned}\sigma_k\pi_k(s) = \sigma_{k-1}\pi_{k-1}(s + a_k) + \sigma_{k-1}\{\pi_{k-1}(s + a_k[1 - \overline{\pi}_{kk}(s)])- \\ -\pi_{k-1}(s + a_k)\}\nu_k(s + a_k[1 - \overline{\pi}_{kk}(s)]) + a_k\pi_{kk}(s),\end{aligned} \tag{3}$$

$$\pi_{kk}(s) = \nu_k(s + a_k[1 - \overline{\pi}_{kk}(s)])\overline{\pi}_{kk}(s), \tag{4}$$

where $h_k(s + a_k - a_k\overline{\pi}_{kk}(s))$ and $\nu_k(s + a_k - a_k\overline{\pi}_{kk}(s))$, for each of the schemes I.J, are determined from certain relations respectively, for $s = s + a_k - a_k\overline{\pi}_{kk}(s)$.

Applying the analysed results implies considerable difficulties, the main one – the complexity of the results, their "incompetence" to use them in applications. The obtained results are expressed in terms of recurrent functional equations, Laplace or Laplace-Stieltjes transforms. To determine, for example, $\Pi_k(t)$, we have to solve the equations (1)-(4), then to take the numerical inversion of Laplace-Stieltjes transforms. Even to determine the average value of busy period – it's not such a simple task, firstly it is necessary to calculate the value of Laplace-Stieltjes transform of the function $\pi_k(s)$ in some points. The overcoming of difficulties lies in elaboration of numerical algorithms and software.

3 Numerical modeling

Example 1 *Let consider a generalized queueing system with DD priority, formed from k queues,* $k = \overline{1,10}$. *The requests arrive according to Poisson flow with parameters:* a_k ={*0.2, 0.5, 0.3, 0.1, 0.8, 0.4, 0.5, 0.6, 0.7, 0.8*}. *The distribution function taken by* $B_k(x)$ *and* $C_k(x)$ *is Exponential, with the following parameters:* b_k ={*0.2, 0.1, 0.8, 0.2, 0.1, 0.1, 0.6, 0.3, 0.2, 0.1*}, c_k ={*0.2, 0.4, 0.9, 0.3, 0.1, 0.1, 0.2, 0.3, 0.2, 0.1*}, $s = 0.1$ *and the set value* $\theta_k = 0.3$.

Table 1. Numerical results of distribution function for k-busy period

k	$h_k(s)$	$\nu_k(s)$	$\pi_k(s)$
1	0.500000	0.800000	0.070265
2	0.580376	0.614390	0.290623
3	0.154268	0.000000	0.148270
4	0.091700	0.099534	0.002969
5	0.271605	0.053822	0.006230
6	0.857142	0.666666	0.319176
7	0.244746	0.000000	0.204944
8	0.115296	0.000000	0.092162
9	0.050040	0.050810	0.001310
10	0.228612	0.035938	0.005375

4 Conclusion

The main purpose of study of polling models is to determine the performance characteristics of the system. This paper deals with modeling of busy period and auxiliary characteristics for DD discipline. The development of algorithms and elaboration on their basis of software allows solving numerically (1)-(4) and reducing the time for analysis of the real systems.

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Application of the operator-difference method to the simulation of MHD astrophysical problems

Sergey Moiseenko, Gennadii Bisnovatyi-Kogan, Nikolai Ardelyan

Abstract

We represent some features of the application of completely conservative Lagrangian operator-difference numerical scheme on triangular gird of variable structure to the simulation of the magnetohydrodynamical (MHD) astrophysical problems. The application of the Lagrangian grid requires its remapping during the evolution of the fluid flow. There are different ways of interpolation of the grid functions on a remapped grid. We discuss in detail the procedure of conservative remapping of grid functions during grid reconstruction procedure. The scheme described in the paper gave us possibility to simulate different astrophysical problems.

Keywords: numerical methods, operator-difference scheme, magnetohydrodynamics

1 Introduction

Numerical simulations of fluid flows in astrophysics have a number of features such as big gradients of functions, wide range of variations of physical parameters, presence of free boundaries and necessity to take into account a number of physical processes. Another important point is to keep conservation of angular momentum.

One of the suitable methods for numerical simulations of astrophysical problems is a completely conservative operator-difference scheme suggested, developed and studied in papers [1, 2] and references therein.

The scheme was successfully applied to numerical simulation of the collapse problem of cold rapidly rotating protostellar cloud [3] and magnetorotational supernova explosion [4].

2 Brief description of the method

The idea of the used completely conservative operator-difference scheme is to construct grid analogs of basic operators such as *grad*, *div*, *rot* with *good* properties.

We know that the differential operator *grad* is conjugated to the differential operator $-div$, *rot* is conjugated to the $-rot$. The grid analogs of these operators are constructed in the way to keep that property.

The scheme is constructed on the triangular Lagrangian grid of variable structure. The triangular grid consists of cells and knots. Some grid functions are defined in knots (velocity, coordinates, gravitational potential, etc.), some grid functions are cell-functions (they are defined in cells) (density, pressure, temperature, magnetic field).

During the evolution of the flow the Lagrangian grid is 'frozen in' to the matter. Due to the non uniformity of the flow such as non uniform contraction or expansion, presence of vortexes the grid is distorting, one can get triangles with rather sharp or obtuse angles. In such situations the approximation properties of the method can be decreased.

For that reason the grid is restructured at every time step in the regions of the computational domain where it is necessary. The reconstruction of the triangular grid consists of 3 simple procedures: elementary restructuring, addition of the knot in the middle of the connection, removal of the knot (see for details [3]). These three simple procedures allow one not only to improve the quality of the grid, but also to concentrate the triangular grid in the regions of flow where we need higher spatial resolution, or to rarefy the grid minimizing a number of grid points. It means that the grid reconstruction procedure allows one to adapt dynamically the grid.

Grid reconstruction procedure consists of two different stages. At first we change the topology of the grid, improving its quality and/or dynamically adapt it. At the second stage we need to interpolate grid functions, defined in cells to a new grid structure. One of the simplest ways is just to interpolate the grid function to a new structure using any smooth interpolation procedure. That simple approach works satisfactorily for the case when we have smooth grid functions. In the case of strong gradients of functions (e.g. shock waves) simple interpolation leads to the development of nonphysical oscillations in the gas flow.

We suggest to interpolate the grid functions by conditional minimization of specially constructed functionals. For example, to interpolate the density grid function we minimize the following functional:

$$F(\rho_i) = \Sigma(\rho_i - \rho_i^*)^2.$$

under condition $m_{new} = m_{old}$. Here ρ_i – new values of the density grid function, $\rho*$ – interpolated on a new grid structure density grid function m_{old} and m_{new} – mass of reconstructing domain before and after grid reconstruction procedure.

Similar conditional minimization of functionals are used for the calculation of pressure, magnetic field components. For the pressure the condition for the functional is a local conservation of the total energy. For calculation of the new values of the magnetic field components we minimize appropriate function under conditions of conservation of the magnetic energy and magnetic flux simultaneously.

The described way of the interpolation of cell grid function allows one to keep conservation laws on the grid.

Detailed description of interpolation procedure of grid function using conditional minimization of the functionals will be published elsewhere.

3 Conclusion

In the paper we described application of the completely conservative operator-difference method on triangular Lagrangian grid of variable structure to the simulation of some MHD astrophysical problems.

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Input-Output model for Republic of Moldova

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Abstract

The twenty three aggregated brunches of the Moldovan economy are considered. Squared 23x23 direct expenditure coefficients matrix was constructed. Optimization problem dealing with price energy growing impact on the final demand (Gross Domestic Product) has been formulated and solved.

Keywords: input-output table, optimization model, pure industry, final demand, direct costs coefficients matrix.

1 Introduction

W. W. Leontief [1] propounded the scheme of an input-output model. Idea of input-output coefficients (direct yield costs) of a single production industry for other production industries, suggested and implemented by W. W. Leontief is the ground of contemporary input-output model. These models were improved theoretically and practically: transiting from static model to dynamic one with the capital investments lag of one or more years, considering environmental factor, analysing and estimating the world economy's further development, forecasting and forming long-term and medium-term indicative planning of national economy. In this paper optimization model based on the input-output square matrix coefficients is considered. Square 23x23 matrix of input-output coefficients of the Moldovan economy in the period 1996-2014 was constructed. The problem of higher energy prices impact on the final demand has been formulated and solved.

2 Problem statement

The square part of the input-output table has n rows and n columns, and the figure in the i row and j column represents the amount of product from industry i delivered to industry j in a particular calendar year. The result of dividing that quantity by the total output of industry j is a coefficient measuring input per unit of output. In this way the $n \times n$ part of the flow table is converted to an $n \times n$ matrix of coefficients, where the entries in the j column include (when supplemented by the j column of factor inputs per unit of output) all inputs needed to produce one unit of output of industry j. This column of coefficients represents the average technology in use in industry j. For simplicity it is assumed that every industry is pure, so that a single characteristic output is produced using a single average technology. Denote A – the $n \times n$ matrix of interindustry coefficients , x – the $n \times 1$ vector of outputs, y – the likewise $n \times 1$ vector of final deliveries, while F is the $k \times n$ matrix of factor inputs. Mathematical models in input-output economics are reffered to per unit of output (one row for each of k factors) and total factor use is the vector f. Then the basic static input-output model is the following:

$$(I - A)^{-1} y = x, \tag{1}$$

$$f = Fx, \tag{2}$$

where $(I - A)^{-1}$ is the inverse matrix, so called the Leontief inverse. It is also known as the multiplier matrix, because the economy needs to produce a larger amount of a specific good (the amount of final demand for that good), final demand y needs to be multiplied by the obtained x. Equations (1) - (2) comprise the basic static input-output model. Much attention is given to conditions that guarantee the multiplier matrix is strictly positive. Such conditions make sense because basic economic logic requires that an increase $\Delta y > 0$ in final demand in equation (1) results in an increase $\Delta x > 0$ in total output. If the matrix $(I - A)^{-1}$ is not strictly positive, this logic could be violated, i.e. equation (1) always has a solution $x > 0$ for $y > 0$. In fact, the study of equation (1) has led to a number of equivalent statements about A, such as:

1. $(I - A)^{-1} > 0$.
2. $(I - A)^{-1} = I + A + A^2 + A^3 + \dots$, that is, the series $\sum A^k$ is convergent.
3. All successive principal minors of $(I - A)^{-1}$ are positive.

4. There exists a choice of units such as all row sums or all column sums of A are smaller than 1.
5. A has a dominant eigenvalue λ, where $0 < \lambda < 1$.
6. The dominant eigenvalue λ of A gets larger, if one element of A is increased, and λ gets smaller, if one element of A is decreased.

Statement 2 says that output $x = y + Ay + A(Ay) + \ldots$. So the quantity y should be produced, plus Ay, which is the vector of input to produce y, etc. Statement 3 is the well-known Hawkins-Simon condition, which assures that each subsystem is productive, i.e., each subgroup of industries i, j, k, requires less input from the economic system than it produces in terms of outputs. According to statement 4, the Brauer-Solow condition, value added in each sector is positive in coefficient matrices derived from input-output tables in (nominal) money values. That is, units for measuring physical output are such as each one costs one monetary unit (thus, if the output unit is lei, the unit price is 1.0 by definition). Assuming that the matrix describes a viable economy, this property assures that if output is measured in any chosen physical units, there exists a set of prices such as each industry has a positive value added (i.e., revenue left to pay for factor inputs). The dominant eigenvalue λ is a measure of the size of intermediate outputs produced in the economy in relation to total production. In other words, λ indicates the net surplus of an economy in the sense that the larger λ is (within the statement 5), the smaller the net output will be. The surplus so defined can be consumed, invested for growth, devoted to environmental protection, etc. Statement 6 is useful for interpreting the role of technological change. Earlier interbranch models, based on the input-output tables for economy of the Republic of Moldova in the period 1998-2004, were examined in [2]. In present article, using the input-output model described above, we tried to solve the following problem. Republic of Moldova imported all kinds of electric and thermic energy therefore its economy is very sensitive to energy prices movement, because these prices have a great impact over all economy, especially on goods and services. The energy industry, being a monopoly, determines prices by itself, in such a manner affecting economy as a whole. The principal goal of this research is to determine how prices in energy sector influence other industries. In this respect input-output table will be used to formulate optimization problem restricted by input-output constraints.

3 Optimization Input-Output model

Now, having elaborated input-output tables for 23 aggregated industries of Moldovan economy in the period 1996-2014, let's formulate the optimization model. Suppose, that energy prices grow at 1,5, i.e. energetic technological column growth on 1,5. How demand vector (i.e. Gross Domestic Product) will be modified in such circumstances? Given the vector of output x, the problem is to maximise $\sum y_i$ restricted by $(I-A)^{-1}y = x$ or

$$\max D = \sum y_i, \tag{3}$$

under the following restrictions:

$$(I-A)^{-1}y = x \tag{4}$$

This optimisation problem, with known output for 23 industries and input-output table for 2014 year, was solved using the Solver application. Calculations effectuated at different rate (0,5;1,0; 1,5; etc.) have demonstrated that final demand diminishes together with energy prices growing. And the conclusion is the following: growth in prices should be done cautiously.

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Wolfram Mathematica as an environment for solving concave network transportation problems

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Abstract

In this work, we consider a transportation problem on a network with concave cost functions and constrained flows on arcs and expose an approach to its solving via Wolfram language algorithm implementation in Wolfram Mathematica System. Our original results are compared with results obtained by applying built-in Wolfram Language functions on a family of test problems.

Keywords: network transportation problem, optimal solution, concave function.

1 Introduction

The concept of transport network may be used to model various economic processes to obtain minimal cost programs for commodity transportation from sources to destinations knowing the available quantities and demands.

We describe the problem of network transportation for which the quantities of commodity transported through each arc is constrained both from above and bottom, and the costs of transportation associated with arcs are defined by linear-concave functions on arc flows.

2 Problem formulation

Let us consider the network transportation problem described by the graph:

$$= (V, E),\ |V| = n,\ |E| = m.$$

A real function of production and consumption $q = V \to R$ is defined on the finite set of its vertices V. Linear-concave cost functions $\varphi_e(x_e)$ are defined for each arc flows.

We need to determine such a flow x^* that minimizes nonlinear objective function $F(x) = \sum_{e \in E} \varphi_e(x_e)$.

2.1 Standard network

The quantity $p(v_0)$ of commodity available for the source v_0 coincides with the required demand $p(v_t)$ of destination v_t in commodity units.

It is required to solve the nonlinear optimization problem:

$$F(x^*) = \min_{x \in X} F(x),$$

where X is a set of admissible flows on described by the following system:

$$\sum_{e \in E^+(v)} x(e) - \sum_{e \in E^-(v)} x(e) = \begin{cases} -p(v) & v = v_0 \\ 0, & V/\{v_0, v_t\} \\ p(v), & v = v_t \end{cases}$$

with both non-negativity constraints and constraints on the transportation capacities of arcs $l(e) \leq x(e) \leq u(e)$, for all $e \in E$.

2.2 Network with one source and several destinations

The quantity $p(v_0)$ of commodity available for the source v_0 coincides with the required demand $\sum_{v \in V_t} p(v)$ for the destinations V_t in commodity units.

We need to solve the nonlinear optimization problem:

$$F(x^*) = \min_{x \in X} F(x),$$

where X is a set of admissible flows on described by the following system:

$$\sum_{e \in E^+(v)} x(e) - \sum_{e \in E^-(v)} x(e) = \begin{cases} -\sum_{v \in V_t} p(v), & v = v_0 \\ 0, & V/V_t \backslash \{v_0\} \\ p(v), & v \in V_t \end{cases}$$

with both non-negativity constraints and constraints on the transportation capacities of arcs $l(e) \leq x(e) \leq u(e)$, for all $e \in E$.

2.3 Network with several sources and destinations

The quantity $\sum_{v \in V_0} p(v)$ of commodity available for the sources V_0 coincides with the required demand $\sum_{v \in V_t} p(v)$ for the destinations V_t in commodity units.

We need to solve the nonlinear optimization problem:

$$F(x^*) = \min_{x \in X} F(x),$$

where X is a set of admissible flows on G described by the following system:

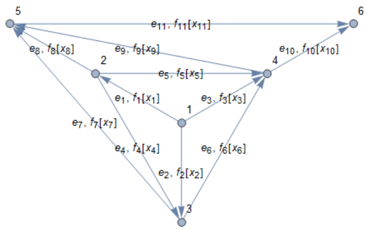
$$\sum_{e\in E^{+}(v)} x(e) - \sum_{e\in E^{-}(v)} x(e) = \begin{cases} -\sum\limits_{v\in V_t} p(v), & v\in V_0 \\ 0, & V/V_t\backslash\{v_0\} \\ \sum\limits_{v\in V_0} p(v), & v\in V_t \end{cases}$$

with both non-negativity constraints and constraints on the transportation capacities of arcs $l(e) \le x(e) \le u(e)$, for all $e \in E$.

3 An algorithm and Wolfram Mathematica program

An original approach to solving the above problems is considered. It may be exposed briefly by the means of the Wolfram language:

```
In[109]:= A = IncidenceMatrix[
```




```
];

{n, m} = Dimensions@A;
b = Table[{0, 0}, n]; b[[1, 1]] = -10; b[[n, 1]] = 10;
X = Array[x, m];
l = {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1};
u = {12, 12, 14, 14, 14, 14, 12, 14, 12, 14, 14};
lu = Table[{l[[i]], u[[i]]}, {i, m}];

γ = {1, 1, 2, 2, 2, 2, 1, 2, 1, 2, 2};
δ = {1, 1, 2, 2, 2, 2, 1, 2, 1, 2, 2};
f[j_Integer, x_] := { γ[[j]] x          x ≤ δ[[j]]
                      γ[[j]] δ[[j]]     x > δ[[j]]
                      0                 True
fd[j_Integer, x_] := { f[j, x] / x   x > 0
                       γ[[j]]        x == 0
                       0             True

X1 = X0 = Values@FindInstance[{A.X == b[[All, 1]], X ≥ 0}, X][[1]];
F0 = Sum[f[j, X0[[j]]], {j, 1, m}];
Z1 = -∞;

i = 0;
While[Not[(Z1 > F0) || (Z1 == F0 && X0 == X1)],
 X0 = X1; F0 = Sum[f[j, X0[[j]]], {j, 1, m}];
 c = Table[fd[j, X0[[j]]], {j, 1, m}];
 X1 = LinearProgramming[c, A, b, lu]; Z1 = Sum[c[[j]] X1[[j]], {j, 1, m}];
 Print[++i];
]
X1
Sum[f[j, X1[[j]]], {j, 1, m}]
```

The above code is exposed for a particular example. But it may be used in the same manner to solve any problem.

4 Conclusion

A series of tests were provided on different test problems to verify efficiency of the approach and program. Our original results have been compared with results obtained by applying built-in Wolfram Language functions on a family of test problems. The approach, algorithm and program proved to be more efficient than built-in Wolfram Mathematica System functions which use numerical algorithms. Our approach, algorithm and program give a more fast execution and solving time than built-in Wolfram Mathematica symbolic functions and methods.

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Numerical method for calculating electrical power in a multiwire transmission line

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Abstract

Transmission line equations are considered for a system consisting of an arbitrary number of electrical conductors. The numerical technique based on finite difference approximation is proposed. First, the original system of differential equations is written in Riemann invariants, and then the resulting equations are approximated according to the finite difference method. In order to obtain the final finite difference scheme with minimal numerical dispersion and dissipation the method of first differential approximation is applied. The novelty lies in obtaining a generalization to the case of an arbitrary number of conductors.

Keywords: transmission line equations, multiwire line, finite difference method, Riemann invariants.

1 Introduction

Interest in the theory of multiple conductor (multiwire), parallel transmission lines extends over the last years because of their numerous applications. The term multiwire transmission line (MTL) typically refers to a set of parallel conductors that serve to transmit electrical signals between two or more points, for example, a source and a load. With some exception, most of the published works has focused on the theory of two parallel, mutually coupled transmission lines. The transmission line equations for a system consisting of an arbitrary number of conductors are derived in [1, 2], starting with Maxwell's equations.

Various methods for solving the transmission line equations for multiwire lines are examined in [2, 3]. The most commonly used numerical techniques for such problems are the finite element method, the finite difference time-domain (FDTD), or the transmission line matrix method [3]. Time-domain differential methods are becoming increasingly popular among the electromagnetic community because of their versatility and their ability to provide simulation results that are intuitively meaningful to circuit designers and microwave engineers. In particular, the FDTD technique offers a mathematically straightforward analysis method, suitable for arbitrary electromagnetic geometries. However, FDTD scheme [3] is sensitive to numerical dispersion and leads to strong non-physical oscillations in numerical solutions in the cases of short circuit and idling. The purpose of the present paper is to obtain reasonably accurate numerical techniques for solving MTL equations with minimal numerical dispersion and dissipation.

2 Mathematical model

Consider the propagation of electromagnetic energy through multiwire three-phase high-voltage transmission line with arbitrary number of conductors. The mathematical formulation of the problem represents the system of partial differential equations known as transmission line equations. The equations are derived from Maxwell equations and for unknown voltage vector $\boldsymbol{u}(x,t)$ and current vector $\boldsymbol{i}(x,t)$ have the following form

$$\boldsymbol{L}\frac{\partial \boldsymbol{i}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial x} + \boldsymbol{R}\boldsymbol{i} = 0 \tag{1}$$

$$\boldsymbol{C}\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{i}}{\partial x} + \boldsymbol{G}\boldsymbol{u} = 0 \tag{2}$$

The domain of the solution of the problem (as well as the domain of the definition for unknown vector functions of the current and voltage) is the rectangle $D = [(x,t) : x \in (0,l), t \in (0,T_{max})]$, where l is the length of the transmission line, T_{max} is the maximal time of calculating for $\boldsymbol{u}(x,t)$ and $\boldsymbol{i}(x,t)$. In an n-wire line the vector functions $\boldsymbol{u}(x,t)$ and $\boldsymbol{i}(x,t)$ have n components each, but $\boldsymbol{L}$, $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{G}$ in Eqs. (1) and (2) are symmetrical matrices $(n \times n)$ of linear inductances, capacitances, wire resistances and conductivities of insulation (vector objects are marked in bold). So we have a system of hyperbolic partial differential Eqs. (1) and (2). To obtain a unique solution, we must add to these equations the initial (when $t = 0$) and boundary (when $x = 0$ and $x = l$) conditions. We assume that at the initial time $t = 0$ there are no voltages and currents in the line

$$\boldsymbol{u}(x,0) = \boldsymbol{i}(x,t) = 0, x \in [0,l]. \tag{3}$$

At the input of the line, at $x = 0$, voltages are given, and at the output for $x = l$ we have a load with resistance $\boldsymbol{R}_s$

$$\boldsymbol{u}(0,t) = \boldsymbol{U}_0(t), \boldsymbol{u}(l,t) = \boldsymbol{R}_s \boldsymbol{i}(l,t). \tag{4}$$

After solving the formulated problem, the active power P is calculated as the average value for the period T of instantaneous power oscillations $p(x,t) = u(x,t)i(x,t)$

$$P(x,t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} u(x,\tau)i(x,\tau)d\tau.$$

Reactive power in the line Q is calculated by the formula

$$Q(x,t) = \frac{1}{\omega T}\int_{t-T/2}^{t+T/2} u(x,\tau)\frac{di(x,\tau)}{d\tau}d\tau =$$

$$= -\frac{1}{\omega T}\int_{t-T/2}^{t+T/2} i(x,\tau)\frac{du(x,\tau)}{d\tau}d\tau.$$

3 Numerical method

In order to solve numerically the formulated problem we propose to apply the finite difference method as follows. To construct the difference scheme, we modify the initial system of Eqs. (1) and (2) and reduce it to a diagonal form, using Riemann invariants. For the transformed system, taking into account the initial and boundary conditions Eqs. (3) and (4), we construct a difference scheme possessing the properties of approximation and stability, and, hence, the convergence. Then using the method of the first differential approximation, we demonstrate that constructed difference scheme has the minimum possible values of dissipation and dispersion terms.

4 Conclusion

The mathematical formulation of the problem for a multiwire power transmission line is studied. The finite difference scheme with minimal values of dissipative and dispersion effects is constructed.

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Piecewise Linear Processes and Their Applications in Finance

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Abstract

We study a piecewise linear renewal process (with doubly jump component), which successively follows independent patterns of two alternating distributions. The financial market model based on this process is studied.

Keywords: renewal process, financial modelling, martingale.

1 Doubly stochastic piecewise linear process

Let $\varepsilon(t) \in \{0, 1\}, t \geq 0$, be a two-state Markov process with the sequence of inter-switching times $\{T_m\}_{m\geq 0}$. Consider the flow of switching instants $T^{+,m} = T_0 + \ldots + T_{m-1},\ m \geq 1,\ T^{+,0} = 0$. Let

$$M(t) = \max\{m \geq 0 \mid T^{+,m} \leq t\},\ t > 0.$$

Let $\tau_{m,n},\ m, n \geq 0$, be the sequence of independent exponentially distributed, $\mathrm{Exp}(\lambda_{m,n})$, random variables, $\lambda_{m,n} > 0$, and $\tau_m^{+,n} = \tau_{m,0} + \ldots + \tau_{m,n-1}, n \geq 1,\ \tau_m^{+,0} = 0$. Let $N_m(t),\ t \geq 0,\ m \geq 0$, be the sequence of (independent) Poisson processes, counting the arrivals of $\tau_m^{+,n},\ n \geq 0$.

We define the double stochastic piecewise linear renewal process by

$$L(t) = \sum_{m=1}^{M(t)} l_{m-1}(T_m) + l_{M(t)}(t - T^{+,M(t)}),$$

where

$$l_m(t) = \int_0^t c_{m,N_m(u)}\mathrm{d}u = \sum_{n=1}^{N_m(t)} c_{m,n-1}\tau_{m,n-1} + c_{m,N_m(t)}(t - \tau_m^{+,N_m(t)}).$$

Process L is supplied with two jump components, described by the compound Poisson processes: $r(t) = \sum_{m=0}^{M(t)-1} r_m(T_m) + r_{M(t)}(t - T^{+,M(t)})$, accompanying each velocity change, where $r_m(t) = \int_0^t r_{m,N_m(u)}\mathrm{d}N_m(u) = \sum_{n=1}^{N_m(t)} r_{m,n}$, and $R(t) = \sum_{m=1}^{M(t)} R_m(N_m(T_m))$, accompanying the patterns' switchings. Here $c_{m,n}$ are constants, $R_m(n)$ and $r_{m,n}$ are independent random variables, independent of the counting processes N_m and M.

We study the distribution of the sum

$$X(t) = L(t) + r(t) + R(t), \qquad t \geq 0, \tag{1}$$

assuming alternation of patterns,

$$\begin{aligned} \lambda_{2m,n} = \lambda_n^i, \lambda_{2m+1,n} = \lambda_n^{1-i}; \quad & c_{2m,n} = c_n^i, c_{2m+1,n} = c_n^{1-i}; \\ R_{2m}(n) \overset{D}{=} R^i(n), R_{2m+1}(n) \overset{D}{=} R^{1-i}(n); \quad & r_{2m,n} \overset{D}{=} r_n^i, r_{2m+1,n} \overset{D}{=} r_n^{1-i}; \\ m \geq 0, n \geq 0,\ i \in \{0,1\}. & \end{aligned} \tag{2}$$

2 Martingality

Let functions $\alpha_i(t)$ and $a_i(t)$, $t \geq 0$, be defined by

$$\alpha_i(t) := \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L(t) + r(t) \mid \varepsilon(0) = i] = \sum_{n=0}^{\infty}(c_n^i + \lambda_n^i\overline{r_n^i})\pi^i(t;n) \tag{3}$$

and

$$a_i(t) = \mathbb{E}[R^i(N(t)) \mid \varepsilon(0) = i] = \sum_{n=0}^{\infty}\overline{R^i(n)}\pi^i(t;n), \quad i \in \{0,1\}. \tag{4}$$

Here $\overline{r_n^i} = \mathbb{E}[r_n^i]$, $\overline{R^i(n)} = \mathbb{E}[R^i(n)]$, are the expectations of the jump amplitudes and $\pi^i(t;n) = \mathbb{P}\{N_m(t) = n \mid \varepsilon(0) = i\}$, $i \in \{0,1\}$. Assume that the series in (3)-(4) converge.

Theorem 2.1. • *Let* $a_i \neq 0$, $\dfrac{\alpha_i(t)}{a_i(t)} < 0$, $\forall t > 0$, *and the integrals* $\int\limits_0^\infty \frac{\alpha_i(t)}{a_i(t)}\mathrm{d}t$, $i \in \{0,1\}$, *diverge.*

If the alternating distributions of T_m *are defined by the survival functions*

$$\overline{F_i}(t) = \exp\left(\int_0^t \frac{\alpha_i(u)}{a_i(u)}\mathrm{d}u\right), \qquad t \geq 0, \qquad i \in \{0,1\},$$

then $X = X(t)$ *is the martingale.*

- *Let* $R^i(n) \equiv 0$ *and* $c_n^i/\overline{r_n^i} < 0$, $\forall n$, $i \in \{0,1\}$.

  If the velocity switchings occur with the intensities $\lambda_n^i = -c_n^i/\overline{r_n^i}$, $i \in \{0,1\}, n \geq 0$, *then* $X = X(t)$ *is the martingale.*

3 Market model

Let process $X = X(t)$ be defined by (1) and condition (2) holds. Assume that the market follows two (alternating) patterns, $\langle c_n^0, r_n^0, \lambda_n^0\rangle_{n\geq 0}$ and $\langle c_n^1, r_n^1, \lambda_n^1\rangle_{n\geq 0}$, during the consecutive elapsed times T_m. The price of risky asset $S(t)$, $t \geq 0$, is given by stochastic exponential of X,

$$S(t) = \mathcal{E}_t(X) = S_0 \exp(L(t)) \prod_{m=1}^{M(t)} (1 + R_m(N_m(T_m))) \times \prod_{n=0}^{N(t)} (1 + r_{m,n}). \tag{5}$$

The dynamics defined by (5) generalises the well-studied jump-telegraph model, [1, 2].

Model (5) can be interpreted as follows. Between time instants $T^{+,m}$ market operates in a usual way. Further, at random times $T^{+,m}$, $m \geq 1$, a strategic investor (or *regulator*) provokes a price impact (of the amplitude $R_m(N_m(T_m))$) accompanied by a pattern's switching. The amplitudes of jumps depend on the regulation policy and on the historical behaviour of the current pattern. Such behaviour of the strategic investor can be interpreted as a price manipulation strategy.

Note that if the regulator does not produce jumps of the asset price, then the market is able to hedge all risks, that is, if $R^i(n) \equiv 0$ and $c_n^i + \lambda_n^i \overline{r_n^i} = 0,\ n \geq 0,\ i \in \{0, 1\}$, then $S(t)$ is the martingale, see Theorem 2.1. Hence, the risk-neutral measure exists.

Let the prices jump on $R^i(n)$ after regulation, and let the elapsed times $T_m,\ m \geq 0$, be exponentially distributed with alternating parameters μ^0 and μ^1. The market is still free of arbitrage, if the jump amplitudes satisfy the inequality

$$\frac{\mu^i \overline{R^i}(n) + c_n^i}{\overline{r_n^i}} < 0, \qquad n \geq 0,\ i \in \{0, 1\}, \tag{6}$$

see Theorem 2.1, and cf [2].

If inequality (6) does not hold, then the risk-neutral measures do not exist. Such policy of the strategic investor could trigger the arbitrage.

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The stability analysis of approximate schemes for the plane fractional order diffusion equation

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Abstract

The evolution equation of fractional order by the space variables is considered in this paper. This equation models the diffusion factor in the process to transport any substance in some medium. An approximate scheme to solve numerically this problem is constructed, using the decomposition principle.

Keywords: diffusion equation, equation of fractional order, approximate scheme.

1 Introduction

The evolution equation simulates a lot of the problems that appear in physique, ecologies, hydrogeology, finance etc. For example, in the mathematical modeling of the problem to transport any substance in atmosphere the main factors are the diffusion process, absorbtion of substance and advection convection process. The classical model of this evolution problem with one space variable uses the usual partial derivatives of first and second order. In recent years many authors use the fractional space derivative to modeling such process. In this article it is considered the same problem with two space variables of the form

$$\frac{\partial\varphi}{\partial t} - d_+(x)\frac{\partial^\alpha\varphi}{\partial_+ x^\alpha} - d_-(x)\frac{\partial^\alpha\varphi}{\partial_- x^\alpha} - d_+(y)\frac{\partial^\alpha\varphi}{\partial_+ y^\alpha} - d_-(y)\frac{\partial^\alpha\varphi}{\partial_- y^\alpha} = f(x,y,t),$$

$$\varphi(x,y,0) = s(x,y), \qquad \varphi(x,y,t) = 0 \qquad on \qquad the \qquad \partial D, \qquad (1)$$

in the domain $D = [0, a] \times [0, b]$ with the boundary ∂D and the time interval $[0, T]$, where $1 < \alpha \leq 2, 0 < x < a, 0 < y < b, 0 \leq t \leq T, d_+(x) \geq 0, d_-(x) \geq 0$. The left-hand (+) and the right-hand (-) fractional derivatives of order α in (1) are defined by Riemann-Liouville formulas

$$\frac{\partial^\alpha \varphi}{\partial_+ x^\alpha} = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_0^x \frac{\varphi(\xi, y, t) d\xi}{(x-\xi)^{\alpha+1-m}},$$

$$\frac{\partial^\alpha \varphi}{\partial_- x^\alpha} = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_x^a \frac{\varphi(\xi, y, t) d\xi}{(\xi - x)^{\alpha+1-m}}, \tag{2}$$

where m is a less integer such that $m - 1 < \alpha \leq m$. The analogous formula holds for the fractional derivative by variable y.

2 The decomposition principle in construction of the weighted approximate schemes

To construct an approximate scheme for (1) let τ be the time step, $t_n = n\tau$ and h is a space step of grid, $x_i = ih, y_k = kh\ \ i, k = 0, 1, 2, \ldots, M$. Let $\varphi_{i,k}^n$ be the numerical approximate value of $\varphi(x_i, y_k, t_n)$. Using the notations

$$A_x = -d_+(x)\frac{\partial^\alpha \varphi}{\partial_+ x^\alpha} - d_-(x)\frac{\partial^\alpha \varphi}{\partial_- x^\alpha}, \quad A_y = -d_+(y)\frac{\partial^\alpha \varphi}{\partial_+ y^\alpha} - d_-(y)\frac{\partial^\alpha \varphi}{\partial_- y^\alpha} \tag{3}$$

the equation (1) can be written in the form

$$\frac{\partial \varphi}{\partial t} + A_x \varphi + A_y \varphi = f. \tag{4}$$

The operators A_x, A_y are defined in the space of functions $\varphi(x, y, t)$ that satisfies the initial and boundary conditions of the problem (1) and for any fixed value of t this function belongs to $L^2(D)$. Also we consider that the operators A_x, A_y are positive defined and are discretized using the following Grunwald formula for the left-hand and right-hand fractional derivatives:

$$\frac{\partial^\alpha \varphi}{\partial_+ x^\alpha} = \frac{1}{h^\alpha} \sum_{m=0}^{i+1} g_m \varphi(x_i - (m-1)h, y_k, t_n) + o(h)$$

$$\frac{\partial^\alpha \varphi}{\partial_- x^\alpha} = \frac{1}{h^\alpha} \sum_{m=0}^{M-i+1} g_m \varphi(x_i + (m-1)h, y_k, t_n) + o(h), \tag{5}$$

and the analogous formulas for the variable y, where

$$g_0 = 1, \qquad g_m = (-1)^m \frac{\alpha(\alpha-1)...(\alpha-m+1)}{m!}, \qquad m = 1, 2, 3, \ldots$$

We denote the corresponding operators defined by the right sides of the equality (5) through Λ_x^+ and Λ_x^-. For the variable y the analogous operators are denoted by Λ_y^+ and Λ_y^- using corresponding equality by the variable y. Thus the operator A_x is approximated by $\Lambda_x = \Lambda_x^+ + \Lambda_x^-$ and the operator A_y is approximated by $\Lambda_y = \Lambda_y^+ + \Lambda_y^-$.

The approximation of the operators Λ_x, Λ_y by the variable t on the interval $[t_{n-1}, t_{n+1}]$ is defined as the weighted average with the parameter r of each operator Λ_x^+ and Λ_x^- at the time points t_{n-1} and t_{n+1}. Thus at the point t_n the value of the operator $\Lambda_x \varphi$ is replaced by the formula

$$\Lambda_x^n \varphi = \frac{d_i^+}{h^\alpha} [r \Lambda_x^+ \varphi(t_{n-1}) + (1-r) \Lambda_x^+ \varphi(t_{n+1})]$$

$$+ \frac{d_i^-}{h^\alpha} [r \Lambda_x^- \varphi(t_{n-1}) + (1-r) \Lambda_x^- \varphi(t_{n+1})]. \tag{6}$$

The operator $\Lambda_y \varphi$ is replaced by the analogous formula. The first order time derivative is discretized using the central finite difference formula. In the next formulas we use the notation $\varphi(x_i, y_k, t_n) = \varphi_{i,k}^n$ and the inferior index will be omitted. According to the decomposition principle, equation (4) with two space variables is approximated on the interval $t_{n-1} \leq t \leq t_{n+1}$ by a system of four equations. All equations of this system has the similar form. We consider, for example, the first equation on time interval $[t_{n-1/2}, t_{n-1}]$

$$\frac{\varphi^{n-\frac{1}{2}} - \varphi^{n-1}}{\tau} + \Lambda_x^n \varphi^{n-\frac{3}{4}} = 0. \tag{7}$$

In detailed form this equation becomes

$$\frac{\varphi_{i,j}^{n-1/2} - \varphi_{i,j}^{n-1}}{\tau} = \frac{d_i^+}{h^\alpha}[r\sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1/2} + (1-r)\sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1}]$$

$$+\frac{d_i^-}{h^\alpha}[r\sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1/2} + (1-r)\sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1}].$$

For the second and third equations of the system (7) the function $f \neq 0$ and is approximated by addition of the the sum $rf_{i,j}^{n-1/2} + (1-r)f_{i,j}^{n-1}$ at the right side of the precedent equality. Through simple transformations we obtain

$$\varphi_{i,j}^{n-1/2} - (1-r)[\xi_i \sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1/2} - \eta_i \sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1/2}] =$$

$$\varphi_{i,j}^{n-1} + r[\xi_i \sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1} + \eta_i \sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1}] \tag{8}$$

for $i = 1, 2, \dots, M-1, \qquad n = 1, 2, \dots, N-1$, where

$$\xi_i = \frac{\tau d_i^+}{h^\alpha}, \qquad \eta_i = \frac{\tau d_i^-}{h^\alpha}.$$

Using the boundary conditions it is obtained the matrix form of (8)

$$[I - (1-r)A]\Phi^{n-1/2} = (I + rA)\Phi^{n-1}, \tag{9}$$

$$\Phi^{n-1/2} = [\varphi_{1,j}^{n-1/2}, \varphi_{2,j}^{n-1/2}, \dots, \varphi_{M-1,j}^{n-1/2}]' \quad \text{and}$$

$$\Phi^{n-1} = [\varphi_{1,j}^{n-1}, \varphi_{2,j}^{n-1}, \dots, \varphi_{M-1,j}^{n-1}]'$$

and the elements of the matrix $A = (a_{i,j}), i, j = 1, 2, \ldots, M-1$ are defined by following formulas:

$$a_{i,j} = \left\{ \begin{array}{cc} (\xi_i + \eta_i) g_1, & j = i, \\ \xi_i g_2 + \eta_i g_0, & j = i-1, \\ \xi_i g_0 + \eta_i g_2, & j = i+1, \\ \xi_i g_{i-j+1}. & j < i-1, \\ \eta_i g_{j-i+1}, & j > i+1. \end{array} \right\}. \tag{10}$$

For approximate scheme (7) the following theorem is true:

Theorem 1. *The order of local error approximation of equation (7) by the system (9) is $O(\tau + h)$ for $0 \le r < \frac{1}{2}$ and this order is $O(\tau^2 + h)$ for the value of $r = \frac{1}{2}$.*

3 Stability of the approximate scheme

In order to prove the stability of scheme (7) (in its matrix form (9)) we first remark that the coefficients ξ_i, η_i are non-negative. Also the Grunwald coefficients g_k satisfy the following properties

$$g_1 = -\alpha, g_k \ge 0 (k > 1), \quad \sum_{k=0}^{\infty} g_k = 0, \sum_{k=0, k \ne 1}^{N} g_k \le \alpha, \quad N = 1, 2, \ldots \tag{11}$$

The stability of scheme (7) is considered in sense of recurrence. In matrix form first equation of this scheme can be written in the form

$$\Phi^{n-1/2} = P \Phi^{n-1} : \textit{with } P = [I - (1-r)A]^{-1}(I + rA),$$

and Φ^0 is obtained from the initial condition of the problem.

Theorem 2. *The approximate scheme (11) of the initial problem is unconditionally stable for $0 \le r \le 1/2$. If $1/2 < r \le 1$, then the scheme (7) is conditionally stable when*

$$(\xi + \eta)\alpha \le \frac{1}{2r-1} \frac{h^\alpha}{\tau},$$

where $\xi = max(\xi_x, \xi_y)$ *and* $\xi_x = max(d_+(x))$, $\eta = max(d_-(x))$ *on the interval* $[0, a]$, $\xi_y = max(d_+(y))$, $\eta = max(d_-(y))$ *on* $[0, b]$.

Proof. According to (10) we have for the diagonal elements of the matrix A $a_{ii} = (\xi_i + \eta_i)g_1 = -(\xi_i + \eta_i)\alpha$. The sum of non-diagonal elements of line i , denoted by R_i can be evaluated as follows

$$R_i = \sum_{k=1,k\neq i}^{M-1} A_{ik} = \sum_{k=0,k\neq 1}^{i} \xi_i g_k + \sum_{k=0,k\neq 1}^{M-i} \eta_i g_k \leq (\xi_i + \eta_i)\alpha.$$

Using Gherschgorin theorem for every eigenvalue λ there exist A_{ii} such that

$$\mid \lambda - A_{ii} \mid \leq R_i, \qquad |\lambda + (\xi_i + \eta_i)\alpha| \leq (\xi_i + \eta_i)\alpha.$$

From the last inequality the real parts of the eigenvalue of matrix A are non-positive. The eigenvalue of matrix P is

$$\lambda_P = \frac{1 + r\lambda}{1 - (1 - r)\lambda}.$$

If $0 \leq r \leq 1/2$ the inequality $\mid \lambda_P \mid \leq 1$ holds for any r, then the spectral radius of the matrix P is not greater than 1, therefore $\| P \| \leq 1$ and the algorithm is stable. If $1/2 \leq r \leq 1$, the inequality $\mid \lambda_P \mid \leq 1$ is established under the condition $(\xi_i + \eta_i)\alpha \leq \frac{1}{2r-1}$. Hence, the scheme (9) is conditionally stable under the restriction

$$(\xi + \eta)\alpha \leq \frac{1}{2r - 1}\frac{h^\alpha}{\tau}.$$

The proof of stability for equations (2)-(4) of (7) is analogous.

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Fuzzy multicriterial optimizations in the transportation problem

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Abstract

In the paper an iterative fuzzy programming approach for solving the multi-objective transportation problem of "bottleneck" type with some imprecise data is developed. Minimizing the worst upper bound to obtain an efficient solution which is close to the best lower bound for each objective function iterative, we find the set of efficient solutions for all time levels.

Keywords: fuzzy programming, fuzzy model, transportation problem, efficient solution.

1 Introduction

It's well known, the increasing of criteria number and imposing of minimal time for realizing the model solution leads only to increasing of solution accuracy for optimal decision making problems. There are many efficient algorithms that solve such models with deterministic data [2]. Since in real life, some parameters are often of fuzzy type, in the proposed work this case is studied.

2 Problem formulation

Because in any optimization model, objective function coefficients have the largest share in the objective function variations, we shall consider these of fuzzy type and develop the next multi- criteria transportation problem of "bottleneck" type with fuzzy costs coefficients:

$$\min Z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{1} x_{ij} \quad \min Z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{2} x_{ij}$$

$$\ldots\ldots\ldots$$

$$\min Z_r = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{r} x_{ij} \quad \min Z_{r+1} = \max_{i,j} \{\, t_{i,j} |\, x_{i,j} > 0\} \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad \forall i = \overline{1,m}, \quad \sum_{i=1}^{m} x_{ij} = b_j, \quad \forall j = \overline{1,n},$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \quad x_{ij} \geq 0 \; for \; all \;\; i \; and \; j,$$

where : $\tilde{c}_{ij}^{k}$, $k = 1, 2 \ldots r, i = 1, 2, \ldots m, j = 1, 2, \ldots n$ are costs or other amounts of fuzzy type, t_{ij} – necessary unit transportation time from source i to destination j, a_i – disposal at source i, b_j – requirement of destination j, x_{ij} – amount transported from source i to destination j.

In the model there may exist the criteria of maximum too, which however does not complicate it.

3 Theoretical analysis of fuzzy cost multi-criteria transportation model

Since the parameters and coefficients of transportation multi-criteria models have real practical significances such as unit prices, unit costs and many other, all of them are interconnected with the same parameter of variation, which can be calculated by applying various statistical methods. We propose to calculate it using the following formula:

$$p_{ij}^{k} = \frac{c_{ij}^{k} - \underline{c}_{ij}^{k}}{\bar{c}_{ij}^{k} - \underline{c}_{ij}^{k}}, \tag{2}$$

where: $\underline{c}_{ij}^{k}$, $\bar{c}_{ij}^{k}$ – are the limit values of variation interval for each cost coefficient c_{ij}^{k}, where: $i=\overline{1,m},\ j=\overline{1,n},\ k=\overline{1,r}$.

Agreeing to the formula (2), the parameters $\{p_{ij}^{k}\}$ can be considered as the probabilistic parameters of belonging for every value of coefficients $\{c_{ij}^{k}\}$ from their corresponding variation intervals.

The main idea of the method that follows, is the simultaneous and interconnected variation of objective functions coefficients. This makes it possible to reduce the model (1) to a set of deterministic models that can be solved by applying the fuzzy techniques [1].

4 Some reasoning and algorithms

Seeing that the model (1) is of multi-criteria type, for its solving usually it builds a set of efficient solutions, known also as Pareto-optimal solutions. Since solving model (1) involves its iterative reducing to some deterministic we should propose firstly the following definitions.

Let us suppose that: $\left(\bar{\mathrm{X}},\bar{\mathrm{T}}\right)$ is one basic solution for the model (1), where: $\bar{\mathrm{T}} = \max\limits_{i,j}\{\bar{t}_{ij}/\bar{x}_{ij}>0\}$ *and* $\overline{X}=\{\overline{x_{ij}}\}$, $i=\overline{1,m}$, $j=\overline{1,n}$ *is one basic solutions for the first* $r-$ *criteria model* (1).

Definition 1. *The basic solution* $\left(\bar{\mathrm{X}},\bar{\mathrm{T}}\right)$ *of the model (1) is a basic efficient one if and only if for any other basic solution* $(X,T)\neq\left(\bar{X},\bar{T}\right)$ *for which exists at least one index* $j_1\in(1,...r)$ *for which the relation* $Z_{j_1}(X)\leq Z_{j_1}\left(\bar{X}\right)$ *is true, there immediately exists another, at least, one index* $\exists j_2\in(1,....r)$, *where* $j_2\neq j_1$, *for which at least, one of the both relations* $Z_{j_2}\left(\bar{X}\right)<Z_{j_2}(X)$ *or* $\bar{T}<T$ *is true. If all of these three inequalities are verified simultaneously with the equal sign, it means that the solution is not unique.*

Definition 2. *The basic solution* $\left(\bar{\mathrm{X}},\bar{\mathrm{T}}\right)$ *of the model (1) is one of the optimal (best) compromise solution for a certain time* $\bar{T}$, *if the solution* $\bar{\mathrm{X}}$ *is located most closely to the optimal solutions of each criterion.*

In order to solve deterministic model (1) we can use the ***fuzzy technique*** [1] and iteratively solve the deterministic model (3) for the best - L_k and the worst $U_k$values of k-criterion.

Max λ *in the same availability conditions as in* (1) *and:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij} + \lambda \cdot (U_k - L_k) \leq U_k, \quad k = \overline{1,r}, \tag{3}$$

By iterative applying the fuzzy technique for each increasing time level, we could get the set of all its optimal compromise solutions.

5 Conclusion

By applying the hypothesis about the interconnection and similarly variation of the model's objective functions coefficients, we reduce the model (1) to several models of deterministic type, each of which may be solved using fuzzy technique.

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Finding the set of all Nash equilibria of a polymatrix mixed-strategy game

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Abstract

The method of intersection of best response mapping graphs is applied to determine the Nash equilibrium set of a finite mixed-strategy game. Results of a Wolfram language implementation of the method are presented. Appeared issues are highlighted, too.

Keywords: noncooperative game, polymatrix game, mixed strategy, Nash equilibrium set, best response mapping.

1 Introduction

The problem of all Nash equilibria finding in bimatrix game was considered earlier by Vorob'ev (1958) and Kuhn (1961), but as it is stressed by different researchers (see e.g. Raghavan (2002)), these results have only been of theoretical interest. They where rarely used practically to compute Nash equilibria as well as the results of Mills (1960), Mangasarian (1964), Winkels (1979), Yanovskaya (1968), Howson (1972), Eaves (1973), Mukhamediev (1978), Savani (2006), and Shokrollahi (2017). The first practical algorithm for Nash equilibrium computing was the algorithm proposed by Lemke and Howson (1964). Unfortunately, it doesn't compute Nash equilibrium sets. There are algorithms for polymatrix mixed strategy games too [4, 1].

Currently, the number of publications devoted to the problem of finding the Nash equilibrium set is increasing, see, e.g., bibliography surveys in [2, 3].

In this work, we present the results of a Wolfram Language implementation of the method of intersection of best response mapping graphs in polymatrix mixed-strategy games.

2 Problem formulation

The Nash equilibrium set is determined as the intersection of best response mapping graphs [4, 5]. This idea yields a natural method for Nash equilibrium set computing in mixed extensions of two-player $m \times n$ games and n-player $m_1 \times m_2 \times \cdots \times m_n$ games.

Consider a noncooperative finite strategic game:

$$\Gamma = \langle N, \{S_p\}_{p \in N}, \{a^p_{\mathbf{s}} = a^p_{s_1 s_2 \ldots s_n}\}_{p \in N} \rangle,$$

where

- $N = \{1, 2, ..., n\} \subset \mathbb{N}$ is a set of players,
- $S_p = \{1, 2, \ldots, m_p\} \subset \mathbb{N}$ is a set of (pure) strategies of the player $p \in N$,
- $\#S_p = m_p < +\infty$, $p \in N$,
- $a^p_{\mathbf{s}} = a^p_{s_1 s_2 \ldots s_n} : S \to \mathbb{R}$ is a player's $p \in N$ payoff function,
- $S = \underset{p \in N}{\times} S_p$ is the set of profiles.

A mixed extension of Γ or a mixed-strategy game $\widetilde{\Gamma}$ is

$$\widetilde{\Gamma} = \langle X_p, f_p(\mathbf{x}), p \in N \rangle,$$

where

- $$\begin{aligned} f_p(\mathbf{x}) &= \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \cdots \sum_{s_n=1}^{m_n} a^p_{s_1 s_2 \ldots s_n} x^1_{s_1} x^2_{s_2} \ldots x^n_{s_n} \\ &= \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \cdots \sum_{s_n=1}^{m_n} a^p_{\mathbf{s}} \prod_{p=1}^{n} x^p_{s_p} \end{aligned}$$

  is the payoff function of the p^{th} player;

- $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n) \in X = \underset{p \in N}{\times} X_p \subset \mathbb{R}^m$ is a global profile;
- $m = m_1 + m_2 + \cdots + m_n$ is the profile space dimension;
- $X_p = \left\{ \mathbf{x}^p = (x_1^p, \ldots, x_{m_p}^p) : \begin{array}{c} x_1^p + \cdots + x_{m_p}^p = 1, \\ x_1^p \geq 0, \ldots, x_{m_p}^p \geq 0 \end{array} \right\}$ is the set of mixed strategies of the player $p \in N$.

The problem of finding all Nash equilibria in $\widetilde{\Gamma}$ is considered.

3 Best response mapping graphs intersection

Consider the n-player mixed strategy game $\widetilde{\Gamma} = \langle X_p, f_p(\mathbf{x}), p \in N \rangle$. The payoff function of the player p is linear if the strategies of the others are fixed, i.e. the player p has to solve a linear parametric problem

$$f_p\left(\mathbf{x}^p, \mathbf{x}^{-p}\right) \to \max, \quad \mathbf{x}^p \in X_p, \quad p = 1, \ldots, n,$$

with the parameter vector $x^{-p} \in X^{-p}$.

Theorem 3.1. *The set of Nash equilibria in polymatrix mixed-strategy game is equal to*

$$NES(\widetilde{\Gamma}) = \bigcup_{\substack{i_1 \in U_1,\ I_1 \in \mathcal{P}(U_1 \setminus \{i_1\}) \\ \cdots \\ i_n \in U_n,\ I_n \in \mathcal{P}(U_n \setminus \{i_n\})}} X\left(i_1 I_1 \ldots i_n I_n\right).$$

The proof of the theorem has a constructive nature. It permits to develop on its basis both a general method for Nash equilibrium set computing, and different algorithms based on the method.

The components $X\left(i_1 I_1 \ldots i_n I_n\right)$ are solution sets of systems of multi-linear simultaneous equations. Their solving needs special conceptual and methodological approaches both from the perspectives of multi-linear algebra and algorithmic theory.

The Wolfram Programming Language, which has a symbolic nature by its origin, is a valuable practical tool for the set of Nash equilibria computing and representation.

4 Conclusion

The symbolic and numerical strength of the Mathematica System and the Wolfram Language permits to construct a package oriented on finding the set of all Nash equilibria in polymatrix mixed-strategy games.

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On the control of a nonlinear beam

Kenan Yildirim

Abstract

In this paper, optimal vibration control of a nonlinear beam is investigated by means of maximum principle.

Keywords: Nonlinear Beam, Optimal Control, Vibration, Maximum Principle.

1 Mathematical Formulation of the Problem

In this study, we consider the nonlinear partial differential equation[2]

$$w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{txxxx} + [g(w_{xx})]_{xx} = f(x,t), \tag{1}$$

where w is the transversal displacement, $x \in (0,\ell)$ is the space variable, ℓ is the length of the beam, $t \in (0,t_f)$ is the time variable, t_f is the terminal time, $\kappa_1 > 0$ and $\kappa_2 > 0$ are constants, $g(w) = O(w^{1+\theta})$ is the nonlinear term and θ is a positive integer, f is the control function to be determined optimally. Eq.(1) is subject to the following boundary conditions,

$$w(0,t) = 0, \quad w(\ell,t) = 0, \quad w_x(0,t) = 0, \quad w_x(\ell,t) = 0 \tag{2}$$

also following initial conditions;

$$w(x,0) = w_0(x) \in H_0^2(0,\ell), \quad w_t(x,0) = w_1(x) \in L^2(0,\ell). \tag{3}$$

In [2], a weak solution, which is global, for Eq.(1) is presented under some assumptions on the nonlinear term.

2 Optimal Control Problem and Maximum Principle

The aim of the optimal control problem is to determine an optimum function $f(x,t)$ to minimize the performance index functional of the beam at t_f with the minimum expenditure of the control. Therefore, performance index functional is defined by the weighted dynamic response of the beam and the expenditure of the control over $(0,t_f)$ as follows;

$$\mathcal{J}(f(x,t)) = \int_0^{\ell} [\mu_1 w^2(x,t_f) + \mu_2 w_t^2(x,t_f)]dx + \int_0^{t_f}\int_0^{\ell} \mu_3 f^2(x,t)dxdt, \quad (4)$$

where $\mu_1, \mu_2 \geq 0, \quad \mu_1 + \mu_2 \neq 0 \quad$ and $\quad \mu_3 > 0$ are weighting constants. The first integral in Eq.(4) is the modified dynamic response of the beam and the last integral represents the measure of the total control expense that accumulates over $(0,t_f)$. The optimal control of a nonlinear beam is expressed as

$$\mathcal{J}(f^{\circ}(x,t)) = \min_{f\in L^2(0,t_f:V^*)} \mathcal{J}(f(x,t)) \quad (5)$$

subject to the Eqs.(1)-(3). In order to achieve the maximum principle, let us introduce an adjoint variable $\nu(x,t)$ satisfying the following equation

$$\nu_{tt} + \kappa_1 \nu_{xxxx} - \kappa_2 \nu_{txxxx} = 0 \quad (6)$$

and subjects to the following boundary conditions

$$\nu(x,t) = \nu_x(x,t) = 0 \quad \text{at} \quad x = 0, \ell \quad (7)$$

and terminal conditions $\quad$ at $\quad t = t_f$

$$\nu_t(x,t) - \kappa_2 \nu_{xxxx}(x,t) = -2\mu_1 w(x,t), \quad \nu(x,t) = 2\mu_2 w_t(x,t). \quad (8)$$

A maximum principle in terms of Hamiltonian functional is derived as a necessary condition for the optimal control function. It is proved in [1] that under some convexity assumptions, which are satisfied by Eq.(4), on performance index function, maximum principle is also the sufficient condition for the optimal control function. Then, the maximum principle can be given as follows:

Theorem 1. *(Maximum principle) The maximization problem states that if*

$$\mathcal{H}[t;\nu^\circ, f^\circ(x,t)] = \max_{f\in L^2(0,t_f:V^*)} \mathcal{H}[t;\nu, f(x,t)] \tag{9}$$

in which $\nu = \nu(x,t)$ satisfies the adjoint system given by Eqs.(6)-(8) *and the Hamiltonian function is defined by*

$$\mathcal{H}[t;\nu, f(x,t)] = -\nu f(x,t) - \mu_3 f^2(x,t) + \nu[g(w_{xx})]_{xx}, \tag{10}$$

then

$$\mathcal{J}[f^\circ(x,t)] \leq \mathcal{J}[f(x,t)], \tag{11}$$

where $f^\circ(x,t)$ is the optimal control function.

Proof. By taking the first variation of the $\mathcal{H}$, control function is obtained optimally as follows;

$$f(x,t) = \frac{-\nu(x,t)}{2\mu_3}. \tag{12}$$

□

3 Conclusion

In this study, optimal control of a nonlinear beam is studied and optimal control function is analytically obtained by means of a maximum principle without linearization of the nonlinear term in the equation of motion.

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Section 5

Computer Science and IT

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Algebraic Laws of Timed Mobile Processes

Bogdan Aman, Gabriel Ciobanu

Abstract

We consider an abstract view of the processes interacting in complex distributed systems, and emphasize the behavioural equivalences between migrating processes with timing constraints and local communication.

Keywords: concurrent processes, distributed systems, timed mobility, behavioural equivalence.

1 Introduction

The general framework is given by a set $\mathcal{P}$ of processes (ranged over by P, Q, ...) by using infinite sets of variables and interacting channel names. Over this set we can define the operation $\mid$ of communication/synchronization between processes as $P \mid Q$. The structure $(\mathcal{P}, \mid, \mathbf{0})$ is a commutative monoid. Over this structure there are defined various equivalences, and so to reduce the space by factorization.

Process calculi represent formalisms used to model distributed and concurrent systems by using labelled transition systems. They allow a high-level description of the concurrent processes and definition of several behavioural equivalences between processes as a step towards some automated tools for the verification of interaction (communication and synchronization) between processes. During the past couple of decades, a number of calculi supporting process mobility were defined and studied, for instance the π-calculus [5]. Various features were introduced to obtain specific formalisms able to describe explicit locations (in distributed π-calculus [4]), explicit migration and timers (in timed distributed π-calculus [2] and timed mobile ambients [1]). Discrete time is a key ingredient in some of these formalisms.

In these calculi, the verification techniques are based on two major tools: temporal logics and behavioural relations. Temporal logics are used to specify the properties that systems have to satisfy, while equivalence relations are used as appropriate abstractions for reduction of state spaces. The relationships between these two tools have been established in [3]. A wide spectrum of observational equivalences can be logically characterized in terms of Hennessy-Milner modal logics.

If an equivalence relation on states consistent with system behaviour is provided prior to exploration of a state space, then a condensed state space can be constructed on-the-fly in which the nodes represent equivalence classes of states. The nodes in a condensed state space are often represented by computing a canonical representative for the corresponds equivalence class. Equivalence checking plays a crucial role in automatic verification of safety properties in concurrent systems. In this paper we study various behavioural equivalences taking into account both timers and locations (for migration).

2 Syntax and Semantics of TiMo

The syntax of TiMo is given below.

Processes	P	$::=$	$a^{\Delta lt}!\langle v\rangle$ then P else P' $\mid$	(output)
			$a^{\Delta lt}?(u)$ then P else P' $\mid$	(input)
			$go^{\Delta lt}\ l$ then P else P' $\mid$	(move)
			$P \mid P'$ $\mid$	(parallel)
			0 $\mid$	(termination)
			$id(v)$ $\mid$	(recursion)
			ⓈP	(stalling)
Located processes	L	$::=$	$l[\![P]\!]$	
Networks	N	$::=$	$L \mid L \mid N$	

As usual, we consider processes up to a structural congruence comprising the commutative monoid laws for $|$ and 0. Intuitively, a process $a^{\Delta lt}!\langle v\rangle$ then P else P' attempts to send a tuple of values v over channel a for lt time units. If successful, it continues as process P; otherwise it continues as process P'. Similarly, $a^{\Delta lt}?(u)$ then P else P' is a process that attempts for lt time units to input a tuple of values and substitute them for the variables u. Mobility is implemented by a process $go^{\Delta lt} l$ then P else P' which moves from the current location to the location given by l after lt time units. Since l can be a variable, its value can be assigned dynamically through the communication with other processes, migration actions support a flexible scheme for the movement of processes from one location to another. Processes are further constructed from the (terminated) process 0 and parallel composition $P \mid P'$. A located process $l[\![P]\!]$ specifies a process P running at location l, and a network is composed out of its components $N \mid N'$. A network N is well-formed if the following hold: there are no free variables in N, and there are no occurrences of the special symbol Ⓢ in N. The set of processes is denoted by $\mathcal{P}$, the set of located processes by $\mathcal{L}$, and the set of networks by $\mathcal{N}$.

3 Timed Bisimulation in TiMo

In what follows, we define various equivalences for processes and networks by considering their temporal behaviour.

Definition 1. *Let $\mathcal{R} \subseteq \mathcal{N} \times \mathcal{N}$ be a binary relation.*

1. *$\mathcal{R}$ is a* strong timed simulation *(ST simulation) if*

$$(N_1, N_2) \in \mathcal{R} \wedge N_1 \xrightarrow{\lambda} N_1' \implies \exists N_2' \in \mathcal{N} : \; N_2 \xrightarrow{\lambda} N_2' \wedge (N_1', N_2') \in \mathcal{R} \, .$$

2. *$\mathcal{R}$ is a* strong timed bisimulation *(ST bisimulation) if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are strong timed simulations.*

3. *The* strong timed bisimilarity *is the union $\sim$ of all ST bisimulations.*

Essentially, the above definition treats timed transitions just as any other transitions. It is easy to check that $\sim$ is an equivalence relation. From the point of view of the evolutionary behaviour of TiMo networks, a crucial result is that strong timed bisimulation can be used to compare the complete computational steps of two systems.

Theorem 1. *Let N_1, N_2 be two networks such that $N_1 \sim N_2$. If $N_1 \stackrel{\Lambda @ l}{\Longrightarrow} N_1'$ then there exists $N_2' \in \mathcal{N}$ s.t. $N_2 \stackrel{\Lambda @ l}{\Longrightarrow} N_2'$ and $N_1' \sim N_2'$.*

Conclusion. In this paper we studied the abstract behavioural equivalences between migrating process in distributed systems in terms of local timers and locations.

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Images processing tools for data measurements from interferograms

Vsevolod Arnaut, Ion Andrieş

Abstract

The imaging interferometry analyzing methods based on software processing of interferograms possess unique potential capabilities in domain of precise measurements. This paper presents description of software tools designed for processing interferograms obtained by imaging interferometric microscopy methods in order to retrieve transverse and longitudinal linear dimensions and optical properties of objects.

Keywords: optical measurements, interferogram processing, linear dimensions, optical properties, software tools.

1 Introduction

The interferograms are obtained by optical equipments with embedded CCD digital camera and in general case are either optical images of the measured objects with an interference raster superimposed on them, or they are holograms obtained by recording the phase distribution carrying information on the linear dimensions and optical properties of objects. To fully realize the unique capabilities of imaging interferometric microscopy methods in precise and detailed characterization of functional nanometric materials strict mathematical methods, algorithms and software tools are necessary.

This paper presents the description of two software graphical tools. The first of them is designed for interferograms processing in order to retrieve the measurement data of thickness and optical parameters of thin nanometric functional films commonly used in photonics. Interferograms are obtained by conventional microinterferometer MII-4 equipped with a digital camera. The second tool simulates the measurements of linear dimensions by processing of objects images overlaid with an interferometric raster with known period *d*. Images are obtained by a special holographic setup.

2 Used technologies

The mentioned above two kinds of measurements which can be performed are axial measurements and lateral measurements (measurements of longitudinal and transverse dimensions in relation to the direction of the light beam propagation).

Graphical data for axial measurements are obtained from interferometric microscope MII-4 equipped by digital camera. The optical scheme of this microinterferometer is a combination of Michelson interferometer and the microscope (Figure 1).

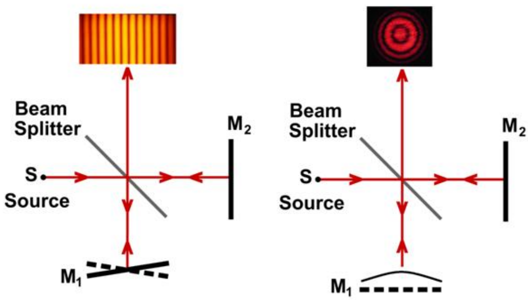


Fig. 1. Microinterferometer optical scheme.

Graphical data represent images of interferograms (Figure 2) obtained from interferometer as a result of direct and reflected beams interference. Because of the thin film cut edge presence (Figure 3), two shifted pictures of interference (Figure 2) are obtained. Measurements performed on the base of images are thickness of opaque thin films, thickness of transparent thin films, refractive index of transparent films. The calculations depends on parameter *b* representing the distance between near fringes and of parameter *c* representing the distance between the fringe and the corresponding shifted one (Figure 2).

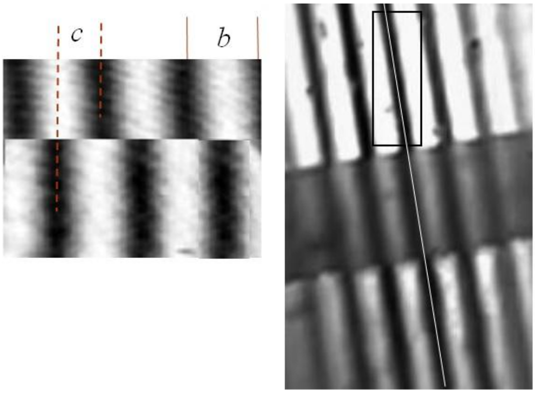


Fig. 2. Examples of interferograms.

Graphical data for lateral measurements also represent interferograms (Figure 4).

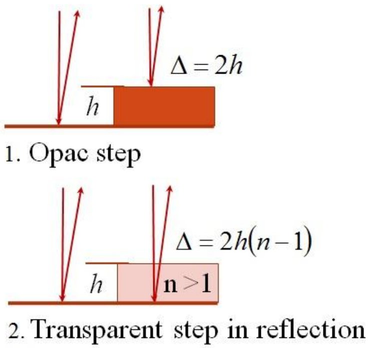


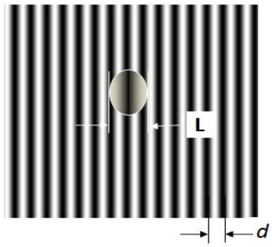
Fig. 3. Cases of opac and transparent films.

3 Graphical tools

Based on the described above algorithms there were developed tools oriented to processing images for retrieval the values of dimensional and optical parameters.

Software tool possesses the following functional possibilities:

- measurement of opaque thin films thickness up to 20-30 nm with a resolution of ~ 5 nm;



$$L = d \cdot \frac{N_L}{N_d}$$

Fig. 4. Algorithm for calculation object size.

- measurement of opaque film thickness with thickness of several wavelengths;
- measurement of transparent film thickness for a given refractive index;
- measurement of refractive index of transparent films at a certain thickness obtained by other measurement methods;
- statistical processing of the measurement results and their storage in a database;
- measurement of objects linear sizes.

4 Conclusion

Compared with conventional layer thickness measuring devices such as profilometers or scanning force microscopes (AFM), this technique provides the full view field of analyzing specimens, is more rapid, noncontact, and does not require complicated specimen preparation. Digital processing of interferograms enables to measure the thickness up to 20 nm with a resolution of ~ 5 nm.

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Virtualized Infrastructure for Integration Heterogeneous Resources

Petru Bogatencov, Nichita Degteariov, Nicolai Iliuha,
Grigorii Horos

Abstract

In the paper there are described directions of distributed and high performance computing (HPC) technologies integration. Analysis of trends in the development of computer technologies, which focused on creating conditions for solving complex problems with high demands of computing resources is presented. The result of these studies is the following conclusion: the main development directions focused on integration of distributed Grid and parallel HPC facilities on the base of virtualization paradigm within integrated Cloud infrastructure in order to expand the range of opportunities for end-users by providing heterogeneous computing resources. Perspectives of utilization of Cloud technologies for integration of Grid and HPC clusters in heterogeneous computer infrastructures that are offering effective resources and end-user interfaces are considered.

Keywords: distributed computing technology, Cloud computing, High Performance Computing, computational clusters.

1 Introduction

In the past years, development of distributed and high-performance computing (HPC) technologies for solving complex tasks with specific demands of computing resources are actively developed, including in Moldova [1]. New areas of works in this direction focused on integration of Grid, HPC and Cloud infrastructures and gain benefit to end users from uniting computational resources of Grid and HPC clusters with effective users interfaces and infrastructure management tools offered by Cloud.

2 Approaches of Heterogeneous Federated Infrastructure realization

These developments are using results of previous projects like the regional project Experimental Deployment of an Integrated Grid and Cloud Enabled Environment in BSEC Countries on the Base of gEclipse (BSEC gEclipseGrid) supported by Black Sea Economic Cooperation Programme (http://www.blacksea-Cloud.net). For this project we selected middleware implementing computing architecture that provides a collaborative, network based model that enables sharing of computing resources: data, applications, storage and computing cycles. The project allowed introducing the general idea of federated Cloud infrastructure, which can offer different solutions for universities, scientific and research communities [2]. The project was focused on implementation approaches to combine the Grid and Cloud resources together as a single enhanced computational power and offers the possibility to use Grid or Cloud resources on demand. As an example, if the user requires parallel computational resources, his jobs submit on the Grid, but if the user needs any specific software or environment to solve some special problem, he can use a dedicated Cloud service or virtual image for that purpose. Fig. 1 shows the skeleton of the suggested platform. The proposed platform made it possible to solve the following problems:

— increasing the effective usage of computational resources;

— providing additional services on demand for scientific and research communities;

— close collaboration between different resources providers.

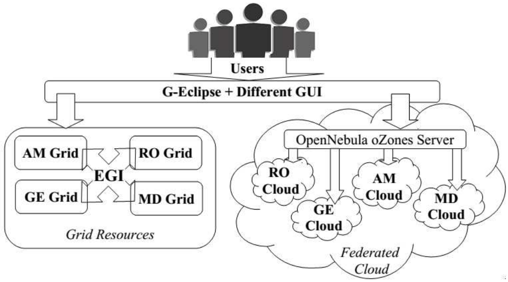

Figure 1. General structure of the proposed heterogeneous regional platform.

3 Integration of Cloud and HPC/GRID resources using OpenStack

Future researches in creation of integrated heterogeneous distributed computing infrastructure were continued within regional project VI-SEEM (VRE for regional Interdisciplinary communities in Southeast Europe and the Eastern Mediterranean) [3]. During preparations for this new project the works were effectuated to unite in one regional infrastructure various distributed computing resources like Grid, HPC, storage and computing Cloud. This is the advantageous step forward, because it will bring us elasticity in resources management, simplify administration and give researchers ability to solve a huge range of computational and visualization problems from small to big complexity in a unified elastic infrastructure. If user needs some kind of general-purpose software or smaller computer resources, that does not require high parallelism; he can use one of available Cloud images. Deployed infrastructure supports almost every mainstream Linux distributions (CentOS, Scientific Linux, Ubuntu, Debian, Fedora, etc.). If he needs more computing cores with more parallelism, he can easily provision a cluster of HPC nodes through OpenStack GUI and add additional nodes if he wants. The main problem is that Cloud and HPC have different principles of resources allocation. In Cloud we run virtual machines but HPC bases on Bare Metal servers (nodes) combined into computing cluster. The approach of provisioning HPC nodes inside virtual machines in OpenStack seems to be the solution, but there are some problems. As high performance is the major constituent of High-Performance Computing, running HPC on virtual machines is not the best solution, because virtualization causes performance drop itself. To achieve better results we must run our HPC nodes on Bare Metal servers. The solution had been found in an OpenStack component called "Ironic". It is an OpenStack development, which provisions bare metal (as opposed to virtual) machines. It may be used independently or as part of an OpenStack Cloud, and integrates with the OpenStack Identity (keystone), Compute (nova), Network (neutron), Image (glance) and Object (swift) services. When the Bare Metal service is appropriately configured with the Compute and Network services, it is possible to provide both virtual and physical machines through the Compute services API [4]. To achieve the initial idea and ensure heterogeneous resources management for HPC, Grid and storage access on the Cloud we re-deployed our RENAM Scientific Cloud (RSC) infrastructure by using OpenStack 13.1.1 Mitaka middleware. Now it consists of one controller node running on VM and three computing nodes (2 for Virtual Machines provisioning and 1 for Bare Metal provisioning). It has in total 24 CPU cores, 48GB of RAM and 2 TB HDD storage and two 1Gbit networks – one for public access and the other for high-throughput interconnectivity between VMs. In RSC digital certificate TERENA SSL CA 3 is installed. Access to RSC resources is provided via https://cloud.renam.md.

4 Federated IdM to access integrated computing infrastructures

To ensure operation of federated mechanism to access distributed computing resources, there were finalized works to realize solutions that allow providing unified access to Cloud infrastructures and be integrated in the creating Research & Educational identity management federations operated within eduGAIN inter-federation authorization & authentication mechanism. The practical results in the area of implementation of federated access to Cloud are based on realization of EGI-Inspire AAI Cloud Pilot project Federated Authentication and Authorization Infrastructure (AAI) for services of Research and Educational Networks and other new results obtained during deployment and administration of OpenStack Cloud infrastructure [5].

5 Conclusion

Cloud technologies are spreading amazingly fast and already took the lead in many domains of IT application – Science, Medicine, etc. They still penetrating in new niches – every year more and more supercomputers in the top lists are being powered by OpenStack, rather than traditional HPC approach. The reason is in its flexibility and diversity, combined with "modular design". It has a couple of basic core components and a variety of optional (additional) ones, which are used for creating infrastructure of any grade of complexity, heterogeneous ones that can combine virtual machines and bare metal nodes, making it more and more attractive to HPC and GRID users. Our combined HPC and Cloud RSC infrastructure proved its functionality and reliability; anyway, it cannot be considered as a production-ready, as it does not provide necessary High-Availability backup and redundancy yet. In addition, it cannot boast of huge performance, it is more about proof-of-concept. However, it is a very good playground for studying Cloud and HPC computing, and for training IT specialists and researchers to work on HPC, Grid and Cloud clusters

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On the Bisector of a Pair of Strings

Ivan Budanaev

Abstract

In this paper we present the construction algorithm of all points of the bisector of a pair of strings. We consider the Levenshtein and Hamming distances on the space of strings.

Keywords: Metric, Levenshtein distance, Hamming distance, strings, bisector, optimal parallel decompositions.

1 Introduction

Let A be an alphabet and let $L(A)$ be the set of all finite strings $a_1a_2\dots a_n$ with $a_1, a_2, \dots a_{n-1}, a_n \in A$. Let ε be the empty string. Consider the strings $a_1a_2\dots a_n$ for which $a_i = \varepsilon$ for some $i \leq n$, and denote by $L^*(A)$ all strings of this form. It is obvious that $L(A) \subset L^*(A)$. If $a_i \neq \varepsilon$, for any $i \leq n$ or $n = 1$ and $a_1 = \varepsilon$, the string $a_1a_2\dots a_n$ is called a canonical string. The set

$$supp(a_1a_2\dots a_n) = \{a_1, a_2, \dots, a_n\} \cap A$$

is the support of the string $a_1a_2\dots a_n$.

For the string $a = a_1a_2\dots a_n$ we consider the proper length $l^*(a) = n$, and the length $l(a) = |\{i : a_i \neq \varepsilon\}|$. For two strings $a_1\dots a_n$ and $b_1\dots b_m$, their product (concatenation) is $a_1\dots a_nb_1\dots b_m$.

If $n \geq 2, i < n$, and $a_i = \varepsilon$, then the strings $a_1\dots a_n$ and $a_1\dots a_{i-1}a_{i+1}\dots a_n$ are considered equivalent. In this case any string is equivalent to one unique canonical string. Two strings a and b are called equivalent, denoted $a \sim b$, if a and b are equivalent to the same canonical string. In this case, $L(A)$ becomes a monoid with identity ε, whereas $L^*(A)$ is a semigroup.

Our aim is to study the bisector problem in the space of strings.

2 Parallel decompositions

Fix an alphabet A and $\bar{A}$=$A \cup \{\varepsilon\}$. Consider on $\bar{A}$ the discrete metric ρ, where $\rho(a,b) = 1$ for all distinct $a, b \in \bar{A}$. Define on $L^*(A)$ the generalized *Hamming* distance ρ_H: if $a = a_1a_2 \dots a_n$ and $b = b_1b_2 \dots b_m$, then

$$\rho_H = \Sigma\{\rho(a_i, b_i) : i \leq min\{n,m\}\} + |n - m|.$$

Now let $\rho^*(a,b) = inf\{\rho_H(a', b') : a' \sim a, b' \sim b\}$. In [1] the following assertions were proved:

1. $\rho^*(a,b) = min\{\rho_H(a', b') : a' \sim a, b' \sim b, l^*(a') = l^*(b')\}$.
2. ρ^* is the Levenshtein metric on $L(A)$.
3. ρ^* is a pseudo-metric on $L^*(A)$ with the properties:
   - $\rho^*(a,b) = 0 \iff a \sim b$
   - $\rho^*(a,b) = \rho^*(b,a)$
   - $\rho^*(a,c) \leq \rho^*(a,b) + \rho^*(b,c)$
   - $\rho^*(ac,bc) = \rho^*(ca,cb) = \rho^*(a,b)$.

Definition 1. Let $a, b \in L(A)$. The pair $a', b' \in L^*(A)$ is called:

- *parallel decompositions* of a, b if $l^*(a') = l^*(b')$, $a' \sim a$, $b' \sim b$;
- *optimal parallel decompositions* of a, b if $l^*(a') = l^*(b')$, $a' \sim a$, $b' \sim b$, and $\rho^*(a,b) = \rho_H(a', b')$.

In [1] it was proved that the optimal parallel decompositions of a, b may be of form $a' = v_1u_1v_2u_2 \cdots v_ku_kv_{n+1}, b' = w_1u_1w_2u_2 \cdots w_ku_kw_{n+1}$, where $l^*(v_i) = l^*(w_i)$ for any $i \leq n$. In the case of optimal parallel decompositions we have

$$\rho^*(a,b) = \rho^*(a', b') = \Sigma\{\rho_H(v_i, w_i) : i \leq n+1\}.$$

3 Main Result

The set $B(a,b) = \{x \in L(A) : \rho^*(a,x) = \rho^*(b,x)\}$ is called the bisector of strings $a, b \in L(A)$. Let $B^*(a,b) = \{x \in L^*(A) : \rho^*(a,x) = \rho^*(b,x)\}$. Our aim is to construct equivalent representations of the strings from $B(a,b)$ with respect to the optimal parallel decompositions of a and b.

Theorem 1. *Any optimal parallel decompositions a', b' of two strings a, b generate elements $c \in B^*(a,b)$.*

Proof. We present a proof by construction. Let $a' = v_1 u_1 \cdots v_n u_n v_{n+1}$ and $b' = w_1 u_1 w_2 u_2 \cdots w_n u_n w_{n+1}$ be the optimal parallel decompositions of the strings a and b. We divide the construction process of c into three steps:

Step 1. In the first part of the algorithm we analyze subsequences of type v, w (we drop the index i to simplify notation) of strings a', b', and generate subsequences x of c equidistant to v and w. Assume that $v = y_1 y_2 \dots y_l$ and $w = z_1 z_2 \dots z_l$, where $y_i \neq z_i$ and l is the proper length of v and w. To construct a string x such that $\rho^*(x,v) = \rho^*(x,w) = l-k$ we let $x = x_1 x_2 \dots x_l$, where:

1. $x_i = y_i, i \in I_y = \{1 \leq j_1 < j_2 < \dots < j_k \leq l\}$, or
2. $x_i = z_i, i \in I_z = \{1 \leq j_1 < j_2 < \dots < j_k \leq l\}$ and $i \notin I_y$,or
3. $x_i \in \bar{A} \setminus \{y_i, z_i\}$, $1 \leq i \leq l$, $i \notin I_y$ and $i \notin I_z$.

It is not difficult to compute the cardinal of the set of generated x:

$$|B_v^*| = \Sigma \binom{l}{k} \binom{l-k}{k} (|\bar{A}| - 2)^{l-2k}, \text{ where } 0 \leq k \leq \left[l/2\right].$$

Step 2. In this part of the construction process we analyze the subsequences of type u. Assume that $u = t_1 t_2 \dots t_l$. To construct a string x such that $\rho^*(x,u) = l-k$ with $0 \leq k \leq l$, we let $x = x_1 x_2 \dots x_l$, where:

1. $x_i = t_i, i \in I_t = \{1 \leq j_1 < j_2 < \dots < j_k \leq l\}$, or
2. $x_i \in \bar{A} \setminus \{t_i\}$, $1 \leq i \leq l$, and $i \notin I_t$.

The cardinal of the set of generated x by the above method is :

$$|B_u^*| = \sum_{k=0}^{l} \binom{l}{k} (|\bar{A}| - 1)^{l-k} = |\bar{A}|^l.$$

The above computation shows that all strings of length l on alphabet $\bar{A}$ participate in the construction of bisectors of length l of two identical strings of length l. We now proceed to the last step of the algorithm.

Step 3. Once generating all elements of B_v^* and B_u^* is complete, the bisector elements of $c' \in B^*(a', b')$, are of form $c' = v_1' u_1' v_2' u_2' \cdots v_n' u_n' v_{n+1}'$, where $v_i' \in B_v^*$ and $u' \in B_u^*$. If we denote $L = \{l : l = l^*(v_i), 1 \leq i \leq n+1\}$, then the cardinal of set $B^*(a', b')$ is given by the formula:

$$|B^*(a',b')| = |\bar{A}|^{l^*(a)-\rho^*(a',b')} \prod_{l\in L} \sum_{k=0}^{[l/2]} \binom{l}{k}\binom{l-k}{k}(|\bar{A}|-2)^{l-2k}.$$

This completes the proof of the theorem. □

4 Conclusion

An analysis of the proof of the **Theorem 1** allows for the following formulations:

1. Algorithm from **Thereom 1** constructs all $c' \in B^*(a',b')$, such that $\rho^*(a',b')/2 \leq \rho^*(a',c') = \rho^*(c',b') \leq \rho^*(a',b')$. The minimality condition of $\rho^*(a',b')/2$ is proven by the triangle inequality.
2. Algorithm from **Thereom 1** can be extended to generate $c' \in B^*(a',b')$, such that $\rho^*(a',b')/2 \leq \rho^*(a',c') = \rho^*(c',b')$.
3. Elements $c \in B^*(a,b)$ can be generated by taking the canonical representations of $c' \in B^*(a',b')$.
4. One can prove that any element $c \in B^*(a,b)$ can be constructed using the proposed algorithm.

   The presented algorithm can be used as a new technique to solve the problem of finding the center of a set of strings.

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User Interface to Access Old Romanian Documents

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Abstract

A utility and its interface are described that helps the user to select proper OCR patterns to process Romanian Cyrillic Printings of the 17th century. The utility is included in the tool pack for OCR of the Romanian heritage printed in the Cyrillic script.

Keywords: digitization, optical character recognition, the Romanian language, Cyrillic fonts, decision utility.

1 Introduction

The problem of choosing the best suitable optical character recognition patterns for a 17th century printed text represents a part of a high priority in the OCR process [1].

The main goal was to develop a classification algorithm of the 17th century Romanian Cyrillic printings, and implement it in a software utility. The application goes as an extension to the previously developed Tool Pack for digitization and transliteration of the Romanian cultural heritage. The classification algorithm consists of a set of decision rules which are based on the particularities of the typography (printing press) and the geographical region in which the text was printed.

2 Algorithm

The 17th century printing press used many different fonts for books and documents printing. They can be divided in two groups that are very distinct both in style and character usage. We have to mention that in that period the printed text structure is very close to the handwritten text structure. Fig. 1 shows two pages from two different books printed in the 17th century using distinct fonts.

Figure 1. Two pages printed using different fonts; the most distinct letters are demonstrated below the text.

It is obvious that these two texts have different styles in addition to the fact that there are two different-shaped characters for the same letter “t” and “z”. The “t” character from the first text is printed as standard “**т**”, but in the second text the handwritten form of the same letter is used, namely “***m***”. The same is true for the “z” letter, as it is clear that the writing style differs significantly.

Both images represent a scanned Romanian text of the 17th century. Nevertheless, if we use the user recognition patterns trained for the first image, OCR of the second image will result in a very high error level, making the resulting text unintelligible and useless. The only suitable solution for this problem is to create another recognition pattern set, trained especially for the second font.

In order to successfully apply the OCR pattern set for these images, a software tool is needed through which the user could choose the most suitable patterns for his scanned book or document. We will focus only on 17th century books and documents printed in Romanian Cyrillic Script. The decision-making process is described in the following diagram, which allows a better comprehension of the classification algorithm.

Figure 2. Diagram showing the pattern set classification used at optical character recognition.

The diagram on Fig. 2 shows how the user selects the best recognition pattern set for his document, judging by the historical period, region and typography. As it was previously said, we'll focus only on the 17th century Romanian printings. First of all, the user selects the historical period of the scanned text, which in our example is 17th century. Then user selects the geographical location (region) of the Romanian text, which can be one of the following: **Iași, București, Târgoviște, Belgrad (Alba Iulia), Uniev (Cernăuți), Sas Sebeș, Snagov,** or **Buzău.**

The last step is selection of the most appropriate printing press (typography) in the region, which, for example, in Iași can be one of the following:

1. Tiparul cel Domnesc;
2. Casa Sfintei Mitropolii;
3. Tiparnița Tărâi.

In **București** there were the following typographies:

1. Scaunul Mitropoliii Bucureștenilor;
2. Tipografia Domnească;

Belgrad had the following typographyies:

1. Tipografia Domnească;
2. Mitropolia Belgradului;

The most significant typographies in other regions were:

- **Târgoviște** – Sfânta Mitropolie a Târgoviştii;
- **Uniev** – Sfînta Mănăstire Uniev;
- **Sas Sebeș** – Tipografia Noao;
- **Snagov** – Tipografiia Domnească în Sfânta Mănăstire în Snagov;
- **Buzău** – Tipografiia Domnească, la Episcupiia dela Buzău;

The books and documents printed at these typographies can be OCR-ed using one of the two recognition pattern sets trained previously. Therefore, after the printing press is selected, FineReader will run with the best suitable training set already selected for the user's scanned text.

The utility implementing model selection is written in Java core and requires JRE (Java Runtime Environment) and ABBYY FineReader to be installed.

3 Conclusion

Evaluation and digitization of cultural and historic heritage is one of the main goals in the digital European agenda. Even if the process of heritage digitization needs many problems to be solved that refer to recognition, editing, translation, interpretation, it most certainly should be done. These problems became more complicated for Romanian as we need to consider the historic period when the source was printed, and we have more than one period.

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Building of P system based tools for diachronic text analysis

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Abstract

This article considers aspects of application of advanced parallel computing techniques to computational linguistics Big Data problems, which spring from creation Web-based corpora on base of digitization of huge collections of historic texts scans. Solving of the majority of linguistic research problems requires per word processing that is the natural task for massive parallel processing provided by unconventional, high performance and their hybrid computing. Testing our approach, we focus on application of P system computing to diachronic analysis of historic texts.

Keywords: computational linguistic, diachronic analysis, P system, web-based corpus

1 Introduction

The proposed research is aimed to support cultural heritage and historical linguistic domains by advanced techniques and tools of modern ICT. The result supposes to be web-based toolkit providing easy access to textual cultural heritage.

The presented research is addressed today challenge of web Big Data high performance processing. Concerning cultural heritage and historical linguistic domains Big Data are obtained by digitization of huge collections of historic texts scans and creation of corresponding Web-based corpora. The linguistic research that requires per word processing is the natural task for massive parallel processing provided by unconventional or high performance computing. The branch of unconventional computing, we apply as instrument, is P system computing [1], bio-inspired formalism mapping functionality of living cell. Being during last decades our research subject, P system computing demonstrated successful solving of various problems of computational linguistic. P system based syllables division algorithm developed by V. Demidova [2] will be applied in the proposed toolkit for lemmatizing, that is the necessary step for processing rich grammar language like Romanian.

For our approach testing we chose diachronic analysis of historic texts, that is one of the most popular today tasks of textual heritage processing. The idea to develop particular framework for diachronic analysis research has now several implementations and is explained in details in work [3]. Although today diachronic analysis is mostly provided for English where grammatical constructions do not change words significantly, among other researched languages there are German [4], Czech [5], French [6], Italian [7]. To apply general text processing methods to languages with specific scripting, their letters are represented by UTF-16 code.

In the presented work use case choice was inspired by our previous research of decyrillization, which converts rare scripted historical Romanian texts to modern Latin writing. The realized research [8] includes scanned historical text digitization to UTF-16-based representation and creation of web-based resources. The developed resources has been successfully applied to manual diachronic analysis [9] of old Romanian texts of particular period.

Summarizing preceding developments of both domains: P system based linguistic problem solutions and textual cultural heritage preservation - we propose web-based researchers support toolkit with engine working in parallel.

2 Proposed toolkit background and architecture

As it was mentioned in introduction, application of P system computing gives the advantages at problem parallelization stage. General steps of analysis of texts in large corpora certainly include making clusters of neighbor words by corresponding criterion. Such clustering can be done in parallel, so researchers apply clustering algorithms based on modern techniques including neutral networks, genetic and swarm branches of unconventional computing. P system branch also has been applied for clustering since 2005 [10] using various types, but the latest research [11] focus on tissue-like P system. In our previous research we applied tissue P system to medical imaging problems solving [12]. For testing these solutions, corresponding simulators were implemented by both sequential and parallel tools. We intend to adapt these simulators for word clustering.

Another stage, at which P system based algorithms can be applied, is lemmatization - reducing words to their lemmas. This preprocessing has to be done before text analysis because wordforms mostly have the same semantic. P systems based solutions of computational linguistics tasks, developed in the frame of our previous research projects include: mentioned in the introduction syllables division, search in strings, generation of possible prefixes. From elements of these algorithms we intend to build word processing library for proposal toolkit.

Basing on the analysis above, the toolkit architecture supposes to consist of following components:

(1) obtaining texts from corpus – standard Python tools;

(2) pre-processing tools for lemmatization and normalization of selected texts – will be developed as parallel processing algorithms;

(3) diachronic analysis tools – will be complex combination of methods related to English but suitable for our goals and methods considering rich derivations of Romanian, all steps, which can be implemented as parallel algorithms, will be developed applying P system;

(4) results visualization tools - existing ones can be applied directly.

3 Conclusion

This work presents background and architecture of historical linguistic researchers support web-based toolkit with parallel engine applied HPC implementation of P system computing. Toolkit architecture was developed on the basis of analysis of existing tools, their advantages and problems. The continued development of presented architecture and creation of resulted toolkit will be the subject of further research of our groups.

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Adaptive Application for Complex Systems Modeling

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Abstract

The paper presents a formal system for presentation and measurement of applications adaptability, and describes an original methodology for building adaptive applications from various fields of activity/research, including for computer modeling of complex systems in physics.

Keywords: adaptive application, adaptability criteria, personalized adaptive application, complex system.

1 Introduction

Agent-based models (ABM) represent a relatively new methodology designed to study complex systems whose properties synergistically show the individual states of the component subsystems, and can not be deduced through a simple extrapolation of the evolution of components properties from a lower structural level to the higher one, but represent qualitatively new qualities of self-organization. These models can be also developed with adaptive computer applications.

2 Adaptive Applications

An *adaptive application* (*AA*) integrates: hardware, software, methodical approach, design and organizational tools that perform the general automation aspects of defined classes of problems characterized by unique data processing technology, information processing regimes, and conditions for unique operation of hardware and software (adapted after [1]). *AA* are available on workstations of researchers, connected to computer networks and associated in scientific research laboratories. The *Researcher's application* (*RA*) is the hardware and software of *AA*, designed to solve the researcher's concrete problems.

Applications (*A*) *can demonstrate the following types of adaptabilities*: (1) *adapting the computer network* (*CN*) *by adding/ excluding a CN node* (A_{CN})*;* (2) *improving computer speed (CS) to solve research problems in a certain CN node* (A_{CS})*;* (3) *adding/excluding a scientific research laboratory* (*SRL*) *in the application* (A_{SRL})*;* (4) *adding/ excluding an automatic post work* (*APW*) *in SRL* (A_{APW}); (5) *expanding the power of the application* x_i *to solve a new research problem* $p \in \Pi$ (A_{Π})*;* (6) *building the knowledge base* (*KB*) *for a new CN node* (A_{KB})*;* (7) *building a new CN node* (*NCN* - A_{NCN})*;* (8) *modifying the operating system, system and application software in one or more nodes of the CN* (A_{Soft}).

Adapting an application to the advanced field of application requires the following resources: *of staff* (*S*), *of time* (*T*) and *financial* (*F*). In this context, the dimensions of the adaptability of each application are shown in Table 1.

Table 1. Application adaptability dimensions

		A_{CN}	A_{CS}	A_{SRL}	A_{APW}	A_{Π}	A_{KB}	A_{NCN}	A_{Soft}
		1	2	3	4	5	6	7	8
1	*F*	$A_{F,CN}/a_1$	$A_{F,CS}/a_2$	$A_{F,SRL}/a_3$	$A_{F,APW}/a_4$	$A_{F,\Pi}/a_5$	$A_{F,KB}/a_6$	$A_{F,NCN}/a_7$	$A_{F,Soft}/a_8$
2	*S*	$A_{S,CN}/a_9$	$A_{S,CS}/a_{10}$	$A_{S,SRL}/a_{11}$	$A_{S,APW}/a_{12}$	$A_{S,\Pi}/a_{13}$	$A_{S,KB}/a_{14}$	$A_{S,NCN}/a_{15}$	$A_{S,Soft}/a_{16}$
3	*T*	$A_{T,CN}/a_{17}$	$A_{T,CS}/a_{18}$	$A_{T,SRL}/a_{19}$	$A_{T,APW}/a_{20}$	$A_{T,\Pi}/a_{21}$	$A_{T,KB}/a_{22}$	$A_{T,NCN}/a_{23}$	$A_{T,Soft}/a_{24}$

The degree of adaptability of the AA to the family of research issues (FRI) is characterized by three indicators:

- ✓ the ratio between the average cost of achieving a new issue in the *FRI* in *a specific application SA* using *AA* and the average cost of doing the same problem without using *AA*;
- ✓ the ratio of the average staffing requirement ($man \times days$) to the achievement of a new *FRI* problem in an *SA* using *AA* and the average staffing requirement ($man \times days$) of the same problem without using *AA*;

- ✓ the ratio of the average time needed to achieve a new issue in the *FRI* in an *SA* using *AA* and the average time required to achieve the same problem without using *AA*.

Measuring the degree of adaptability of applications and/or comparing the adaptability degree of different applications uses one, several or all of the adaptability dimensions of the applications integrates 24 adaptability indicators of the applications (see Table 1). This adaptability measurement system is universal. Each indicator represents a dimension for assessing the adaptability of applications.

Adaptive applications offer the following benefits: (1) time diminishing, average staffing and average cost of solving problems in an *FRI* in an *SA*; (2) a high degree of standardization of the *SA*; (3) high quality of *SA* and *AA*, and so on.

Each A_i application demonstrates, according to Table 1, the following adaptabilities: $Adapt(A_i) = \{a_{i,1}, \ ..., \ a_{i,24}\}$. Let the end-user requirements for application adaptability are specified by the set of adaptive requirements *Criteria* = $\{c_j | \ c_j \geq 0; j=1, \ldots, 24\}$.

The *Criteria* reprezents a customized criteria system of the beneficiary to evaluate the adaptability of applications. The end user's interest *Interes*(A_i) in the A_i application, measured by the customized adaptive assessment system, is as follows: $Interes(A_i) = \sum_{j=1}^{24} c_j \times a_{i,j}$.

Thus, each beneficiary can build a customized application adaptability assessment system specifying the adaptability criteria.

3. Life Cycle of the Adaptive Application

The fact that an adaptive application produces additional software qualities for the user needs, requires a technique for building adaptive applications that includes [2]: (1) definition of the *AA* requirements; (2) *application domain* (*AD*) description; (3) building a *formal (axiomatized) theory of AD* developed at the step (2); (4) elaboration of the *AD language* of the formal theory developed at the step (3); (5) specifying *families of the AD problems* required by the beneficiary for computing; (6) building the *AD* computer platform; (7) elaborating the adaptation components of the computer platform developed at the step (6) (*Adapter* and other auxiliary modules); (8) generating customized versions of the adaptive applications by end users applying the components developed at the steps (6) and (7); (9) exploitation, maintaining and developing personalized applications built at the step (9) by the end users.

4. Conclusion

In this paper, an original development methodology for adaptive applications and a system for measuring the adaptability of applications have been exposed. The presented methodology was applied to the development of several adaptive applications, including an adaptive-parametric application for modeling complex systems in physics [3].

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Vitalization of Moldavian Printings (1967–1989)

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Rezumat

In the years 1940–1989 the Romanian printings in the Moldavian Soviet Socialist Republic (MSSR) were issued in the specific Moldavian Cyrillic script. Many of them are of interest not only as material to study language development, but as containing useful or unique information. Moreover, some of these editions keep their actuality till now, and it is very desirable to present them to the contemporary readers. To do this, it is necessary to re-edit such printings in the modern Latin script.

This paper describes in details the revitalization of a mathematical book.

Keywords: computer linguistics, OCR, re-edition of printed heritage.

1 Introduction

The book we present as the object of vitalization is *Numbers and Ideals* by V.Andrunachievici and I.Chitoroagă [1]. It was published in Chisinau in Romanian in the Moldavian Cyrillic script (1979).

This book covers some gaps between high school and university courses in mathematics. It is useful for scholars presenting the golden opportunity to widen their horizon in mathematics. With full and strict proofs, the book presents axioms for natural numbers, numerical rings and fields, theory of divisibility in the integer numerical fields, and further generalization of divisibility for quadratic rings and ideals.

The book was selected for re-editing in the Romanian Latin script on the occasion of the centenary of the birth of Vladimir Andrunachievici. We are proud to introduce this text into the modern Romanian cultural space.

2 Digitization, Transliteration, and Text Formatting

The book was scanned to PDF file of images in gray with 600 DPI.

OCR by ABBYY Finereader (AFR) was repeated twice. For the first OCR, we used a small spelling dictionary in the Moldavian Cyrillic script (MC).

The result of the first OCR was used to complete the spelling dictionary. The recognized text was exported from MS Word as the Unicode text. Then it was opened in Notepad++ and broken into words (approx. 73.900).

Approx. 1820 of these words had "-" at their end. Almost all of these appeared because of undetected hyphenation, the situation that can be solved by AFR with the more appropriate word list. Each word of the kind was concatenated with the following word, with several exceptions.

Then the word sequence was restructured in the word list with the "Sort unique" option of the TextFX plug-in for Notepad++. This word list contained approx. 2500 words, including approx. 60 proper nouns.

Then we transliterated [3] the word list to the modern Romanian Latin script (MRL) and introduced the MRL variant in the line with the MC variant of the word. The result was checked against a big (600.000 words) list in MRL that we had created for ELRR [cite, URL]. After merging we got two lists of words in the book in both scripts: that of "good" words and that of "bad" words (approx. 250).

Most of "bad" words appeared due to difficulties in transliteration MC→MRL. The most frequent problem was conversion of **я**→**ia** instead of **я**→**ea** (approx. 160). The "bad" words were checked manually, with the corresponding corrections of the lists.

Now we deleted MRL variants getting a word list (approx. 2320). With this new word list uploaded to AFR, we repeated OCR. Due to the satisfactory printing quality and the newly obtained spelling dictionary, we got less than one OCR error in a page in the text.

Now we transliterated the book text into MRL. The 160 words that were transliterated incorrectly when we worked with the word list were included in the dictionary of exceptions of the transliteration utility to avoid at least these errors.

The book contains a lot of formulas, two small tables, and three diagrams (images). Most of formulas were not recognized except those containing only letters and simple signs like +.

To guarantee the proper quality of formulas, the book should be prepared for printing using LaTeX.

As to the text, it is enough to check it with the MS Word Romanian spelling checker, export it, and adapt its formatting. The main volume of work is restoration of formulas and diagrams: each of them should be rewritten using LaTeX.

As the LaTeX book original is prepared and thoroughly checked, the text pages should be printed on the special parchment in mirrored view. The book is published by the typography from this parchment. The cover is prepared separately.

3 Cyrillic Resource for OCR

To recognize the Moldavian Cyrillic texts with certainty, we need the corresponding spelling word list. We reported above how we worked without it, but with the unified big word list (Moldavian Cyrillic resource) our task became much simpler.

It is not necessary to type all these words manually as we did for the word-lemmas in MRL. Another method seems more attractive: the reverse transliteration of the lexicon in MRL to MC. From three existing lexicons in MRL [3–5], we selected the UAIC lexicon [4], which contains approx. 1 mln. entries, is better structured, and is accompanied with the morpho-syntactic data (MSD tag).

An inconvenience appears in the UIAC lexicon: it contains words from sources of different years, and keeps the original orthography. During these years, use of letters ȃ and ȋ was different. Therefore, some words should be additionally rewritten in the modern orthography.

The transliteration MRL $\to$ MC is performed applying the sequence of filters to the given text [2]. In the order of application, we use: prefix filters; suffix filters; diphthong and triphthong filters; final (letter-to-letter) filters. We added some new filters to solve this ȃ/ȋinconvenience.

Using MSD tag, we deleted from the regular dictionary proper noun, foreign words and abbreviations. At the transliteration these words are processed through the dictionary of exclusions.

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Concept of health self-assessment informational tools for preventable strokes

Svetlana Cojocaru, Constantin Gaindric, Olga Popcova, Iulian Secrieru

Abstract

According to World Health Organization, stroke is one of the leading causes of morbidity. Besides that, stroke is the first cause of adult neurological disability. This paper describes the concept of informational tools for health self-assessment and management in the base of controllable (modifiable) stroke risk factors. Research and development carried out under this concept will help to prevent stroke.

Keywords: information tools, preventable strokes, modifiable risk factors.

1. Introduction

Stroke is one of the leading causes of morbidity and one of the most important problems of modern medicine [1]. Importance of the problem is conditioned by major social and economic impact determined by the corresponding pathologies. According to World Health Organization, stroke is a major cause of death in the developed countries as well.

Immediate mortality rate is far above the medium and around 20% of stroke patients die within 30 days. It also had a tendency to affect young persons of working capacity age. Every year 5,6-6 million people in the world develop strokes.

In the Republic of Moldova stroke incidence and mortality rate remains one of leading in Europe. Stroke incidence rate rose from 20.4 in year 2000 to 28.19 in 2008 in 100000 population. Annually in Moldova more than 10123 persons develop stroke. As in the USA and in other countries cerebrovascular diseases in Moldova are placed on the second place in structure of mortality causes (28.95%) and in the general death rate (16.23%). Annual death rate from stroke in Moldova for 2007 was 54 per 100000 people.

According to National Statistic Agency of Moldova, 14% of patients who survive stroke, demand nursing assistance, 59% remain with neurologic deficit, 27% remain with mild disability. And only less than 10% of patients can return to their former activity. Thus stroke remains the main cause of severe disability [2] and request special responsibility from family members, becoming social and economic burden for the society.

On the other hand about 10% of survivors are fully independent within three weeks, growing to nearly 30% by six months. These numbers can be increased by offering a tool that can predict. There are some changes in health status, which, being considered by individuals under stroke risk, could be overcome, excluding thus the disease appearance.

Many people are not aware that stroke is actually preventable and that stroke survivors can live a normal life afterwards. Thereby, the best approach towards reducing the immense burden that stroke places on our society remains prevention.

The percentage of potentially preventable strokes through the control of modifiable risk factors and lifestyle is around 50% [3]. This fact demonstrates not only the importance of the addressed problem in order to improve the quality of medical services using IT technologies, but also shows the great economic potential of this research. Extension of the working capacity period for a person predisposed to stroke, obviously, brings economic benefits.

2. State of the art

A large number of researches in the field of brain stroke made it possible to identify and classify the major risk factors of this dangerous disease.

There are two groups of risk factors: controllable (modifiable) risk factors (High Blood Pressure/Hypertension, Atrial Fibrillation, High Cholesterol, Diabetes mellitus, Carotid stenosis, Smoking, Alcohol Use, Physical Inactivity, Obesity, Nutrition, Drugs, Inflammation) and uncontrollable risk factors (Age, Gender, Ethnics/Race, Family History, Previous Stroke or transient ischemic attack, Fibromuscular Dysplasia, Patent Foramen Ovale). The best way to protect yourself and your family members from the stroke is to assess and understand your personal predisposition to the cerebrovascular accident, as well as to manage effectively the modifiable risks.

Nowadays there are a lot of general views over the IT solutions to support the clinical decision making. All of them are oriented either to a general view, or to a specific disease. Our interest is on the stroke related information systems.

More efficient and tightly integrated systems for stroke care are needed. In 2005, an American Heart Association task force on the development of stroke systems described the fragmentation of stroke care, defined the key components of a stroke system, and recommended methods for encouraging the implementation of stroke systems of care [4]. The task force defined 7 key components of the Stroke Systems of Care Model:

1. Primordial and primary prevention;
2. Community education;
3. Notification and response of emergency medical services;
4. Acute stroke treatment;
5. Subacute stroke treatment and secondary prevention;
6. Rehabilitation;
7. Continuous quality improvement activities.

The first three components are related directly to the stroke prevention.

There is a growing concern about maintaining health indicators. Mobile/sensor technologies are expected to provide real-time information about vital signs and other physiological indicators of one's health and fitness. Monitoring systems based on these technologies are expected to find greater use in such applications as hospitals, home health monitoring, physician’s offices, elderly care facilities, fitness centres and health research studies.

At present there are many devices that easy allow even an inexperienced user to get a number of physiological parameters of his/her body: numerous fitness wristbands, smartwatches, etc. Applications that allow using mobile devices for testing, tracking and sharing are developed for them. To mention just a few of them: application iHealth blood pressure monitor from the iHealth Lab Inc.; Scanadu Scout device to track several parameters simultaneously (including temperature, heart rate, blood oxygenation, respiratory rate, ECG, diastolic/systolic blood pressure) together with a special application for iPhone; health wristband Simband from Samsung, which continuously monitors vital signs such as temperature, blood pressure, pulse, blood glucose levels, etc. and sends the gathered information to the Samsung Architecture for Multimodal Interactions cloud platform.

Besides this, some interactive systems (including web-) were developed, which allow estimating the stroke risk as a consequence of the answers to a set of simple questions [5]. Such systems are useful for verification at long periods of time, but not for a frequent monitoring.

3. Information tools for preventable strokes

Health self-assessment informational tools (as web and mobile applications) for individuals predisposed to preventable strokes are aimed at comprehensible suggesting (in a manner understandable to anyone) individuals predisposed to stroke some advices regarding the actions that should be taken if he/she feels some symptoms, which lead to worsening health (using only data that do not need an extra effort). Applications for information and alarming the danger of worsening could be accessed by individuals predisposed to stroke through the mobile phone.

We assume that the informational tools for health self-assessment and management will be able to work with data collected by devices and applications described above. The constant appearance of new such devices and corresponding applications, their high cost yet, inability to work with them for a certain categories of end-users require informational tools to be sufficiently flexible and independent (as much as possible) from the source of the input data about the patient's condition.

We selected the most important modifiable risk factors (High Blood Pressure/Hypertension, Atrial Fibrillation, High Cholesterol, Smoking, Alcohol Use, Physical Inactivity, Obesity, Nutrition, Drugs) and persons of working-age 35-64 years.

The following four target groups of potential users were identified:

1. Persons uninformed about their own risk of suffering a stroke;
2. Persons with a high degree of concern (given the family history), but who did not assess their own risk;

3. Persons who have assessed and know their own risk, but do not know that stroke is preventable;
4. Persons who know their own risk and want to control the modifiable risk factors.

For each target group specific IT applications with clearly defined goals should be developed, including decision support systems to assess the stroke risk based on modifiable factors by providing necessary recommendations, information about the appropriate clinics and departments. In case of need the results can be sent directly to decision-making healthcare professionals.

To achieve the main goal the following basic tasks were determined:

- Identifying, storing, structuring and formalizing medical data and knowledge related to the stroke onset.
- Creating a unified knowledge base on the stroke onset problem, based on data extracted from statistics studies (evidence and precedents), literature and experts opinion.
- Elaborating the methodology for creation of some models, based on predictive personalized knowledge regarding health self-assessment, using an analysis of risk factors, designed for individuals predisposed to stroke.
- Developing a system of structuring target groups of persons predisposed to stroke and creating some specific scenarios and strategies for each group.
- Developing IT applications, specific for the given patient's group and the selected scenario.
- Improving medical service by liquidating barriers and inconveniences existing at present on different levels of cooperation between patient, family doctor and neurologist, by organization and support of an integrated flow of data and knowledge.

4. Conclusion

At the moment, in the world there is no a unique informational tool to assess/self-assess the stroke risk factors, to monitor modifiable risk factors and to manage the treatment prescribed by the doctor. The approach to create a unified database containing knowledge, evidence and precedents concerning cause-effect relationships between risk factors and stroke is novel. Orientation to work with various mobile gadgets, in order to organize an exchange of physiological data, is within the last trends in the development of medical information systems.

Such informational tools could be used as tools for research in the field, and will allow developing predictive models of the patient's health status. As a result, they will help the patient and the specialists in choosing the strategy for monitoring the patient's health status.

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Downloading useful information from Web with the help of a Crawler

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Abstract

The article describes a research of a methodology of Web crawler development. The experiment is done on a news site noi.md which is both Romanian and Russian language. Article presents some statistics on dependence between the volume of downloaded information and processing speed.

Keywords: Internet, web-crawler, Jsoup library, text processing.

1 Introduction

The Internet has superflus of information as references, publicity, comments etc. We have to make a plan, how to release the information from the unnecessary data for the future use [1]. The main aim of this article is to study possibilities of Web Crawler and to develop a program for text downloading from the Internet. This program has also to clean the document from the redundant information.

This article consists of: introduction, web-crawler overview, the general information of types and the use of Crawler, implementation of Web Crawler, application structure and results.

2 Web Crawler overview

Web Crawler is a program for page search by the reference. Its work consists of page downloading, revision for new reference and page indexing [2]. Indexing is used for the quicker search of pages inquired by a user [3]. Crawler has to be effective in work so it has to maintain all changes of information it responds to [4]. Web Crawlers are used by searching systems such as google, yahoo, being for general data search. In addition, we use Web Scraper to find special information.

Web Scraper is the modernization of Web Crawler. It is used for data search such as prices, e-mail addreses, phone numbers, images in database and the Internet [5].

Modern Crawlers are classified by such attributes as autonomy – the task must be implemented without a user's support, adaptability – implementation of different types of tasks with minimal changes, communicative ability and the ability to work together with other Crawlers – implementation of tasks in relation with other Crawlers for high efficiency [6]. If an URL makes an error mistake, the next link from the list must be used.

An effective work of Web Crawlers which can be achieved with the crawlers consists of multiple processes running on different machines connected by a high-speed network [3].

A Crawler can be used for: finding the nessesary information according to the user's request; checking connections between pages (links) and the degree of their complication [7]; downloading web pages for the next processing.

3 Implementation of Web Crawler

The developed Web Crawler works in the *noi.md* site area. As Crawler remains in recourse every time, each link for the next processing is verified for compliance with the pattern:

"^(https?)?://[a-z_/.]+/(news_id)/[0-9]{3,6}$"

We use Jsoup library for page downloading and link processing as this library offers very good API which is clear and has a lot of functions for effective work with data.

For the data storage a database is used and all the information is written in files. We can find each news in two main languages: Romanian and Russian, so for more complete data each news is downloaded in both languages.

Program does not need a large space for data saving as only the text information from each page and links are saved. We need about 15.4 seconds for a page download and extraction of information from it. This is a relative calculation based on 3000 downloaded pages.

Each new link for list of URLs is checked for unique to exclude the possibility of repetitive downloading [3], then the cycle is repeated. The data about the dependences between amount of information downloaded, useful text extraction and the processing speed [8] is presented in Figure1.

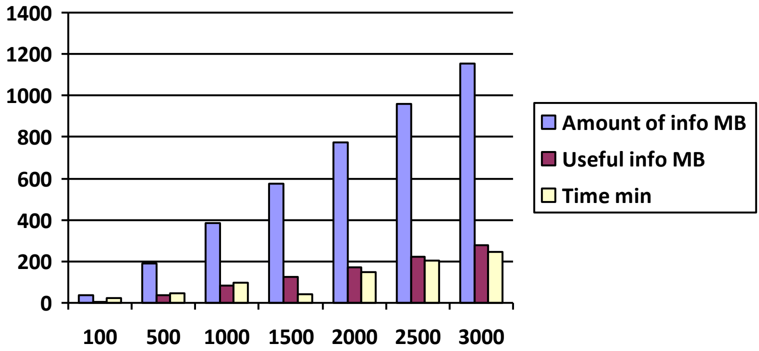

Figure 1: Program's results

Using results in Figure 1 we can also estimate the utility [9] of information, which is presented as quotient between the final data and the downloaded data [10].

The program had a problem while it was tested. This problem appeared because of links contained “https” at the beginning. We suppose this was a problem of Jsoup because it could not download the information from such links. The problem was solved with simple eliminating of the letter „s” from these link. A link without „s” is less secure, so the problem dissapeared.

4 Conclusions

The developed tool is useful in computational linguistic resources creation, which is of great importance in natural language processing applications. Building both large and good quality text corpora is the challenge we face nowadays. The results of this article can become a starting point for data processing of the downloaded text and the creation of text corpora of different domains.

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A linear model for multidimensional Big Data visualization

Vadim Grinshpun

Abstract

The author introduces and analyzes a model that allows organizing visualization of primary linear constructs such as interval, simplex and polygonal lines in multidimensional space.

Keywords: computer science, big data, data visualization, multi-dimensional data, exploratory data research.

1 Introduction

There is a number of well-known methods to visualize multidimensional data. There are Andrews plots, Bergeron's or Wong's model, Zinoviev model as well as Klaft, Barrett and Kleiner-Hartigan, and every one of them introduces their own unique mechanism for data visualization [1]. However, every method has its own limitations, narrowing the field of direct applicability. For instance, the Bergeron's model visualizes the wave lines and the time interval for a single frequency [2]. The main advantage of applying the multidimensional data model is its effectiveness against large sets of data, the main problem is its relative complexity when applies to simple tasks of operational data processing [3].

2 Model Definition

As a basis for the visualization of the multidimensional data we propose to use a linear modification of a multidimensional observation H into two-dimensional curved line $£_{H}(t)$, so H approximates $£_{H}(t)$, with the provable condition that for close values of the dimensional attributes of observations H and X will be shown in close proximity by the graphics reflecting $£_{H}(t)$ и $£_{X}(t)$, in case these values are relatively distant, the graphical lines will appear to be far apart.

For the analysis of the proposed method we will use the most general system of data presentation. Let’s pick a vector H in Pn – a space with finite number of dimensions.

$$\text{H} = (\text{н}0, \text{н}1, \text{н}2, \text{н}3 \ldots . \text{н}n - 1) \in \text{Pn}, \qquad (1)$$

To create the visualization of the vector we have to create a basis for transformation as a set of orthogonal functions {φ (t)} → ∞. Legendre orthogonal polynomials can be applied on a 0 to 1 interval, set of which can be shown as {ʆ (t)} → ∞. In this case the vector H with coordinates (н0, н1, н2, н3….нn-1) ∈ Pn corresponds to the following function:

$$\text{Eн}(t) = \sum_{i=0}^{n-1} \text{HiLi(t)}, \qquad (2)$$

Let’s introduce into the model a second vector:

$$\text{X} = (\text{x}0, \text{x}1, \text{x}2, \text{x}3 \ldots . \text{x}n - 1) \in \text{Pn}, \qquad (3)$$

And its corresponding function:

$$\text{Ex}(t) = \sum_{i=0}^{n-1} \text{HiLi(t)}, \qquad (4)$$

And now we can transform two points H & X from the Pn space, into the graphical view of their representative functions Eн(t) and Ex(t) (Fig. 1).

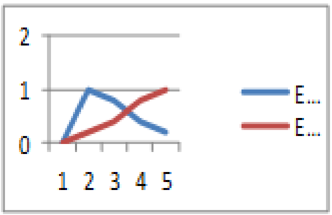

Figure 1. Visualization of H & X from the Pn space.

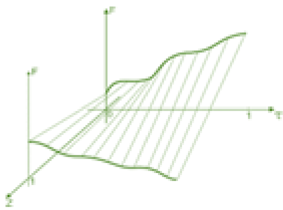

Figure 2. Visualization of smooth surface, corresponding to the HX segment from the Pn space.

When we consider H & X to be vectors, with the beginning located at the beginning of the coordinate system selected for the Pn space – then the relative proximity between all points in the Pn space becomes definitively tied to the graphical representations of their corresponding Eн(t) and Ex(t) functions, with axes values defined as $\text{н}0, \text{н}1, \text{н}2, \text{н}3 \ldots . \text{н}n - 1$. By introducing a variable, we can create an equation:

$$C(c) = (1-c)H + cX = ((1-c)\text{н}0 + cx0, (1-c)\text{н}1 + cx1, \ldots (1-c)\text{н}n - 1 + cxn - 1 \quad (5)$$

From which obviously follows $C(0) = H$ и $C(1) = X$ which can be viewed as a definition of a multidimensional "straight" line connecting H & X in the Pn space, and we can use the expression similar to (5) to represent a multidimensional segment HX

$$HX = (1-c)H + cX, \text{ where } c \in [0,1] \quad (6)$$

Assuming "c" represents the distance in the Pn space and the formula 6 can be shown as the proposed model:

$$E_{HX}(c) = \sum_{i=0}^{n-1}(1-c)\text{н}iLi(t) + cxiLi\,(t), \quad (7)$$

This function has two arguments$\{c, t\}$, which allows us to get a graphical function $E_{HX}(c) = E_{HX}\{c, t\}$ that visually represents the HX segment as a smooth surface. (Fig. 2).

3 Sample Application

To test the model, we will apply it to a sample set of multi-dimensional objects with the following values: *H*1 = {1, 0, 0, 0}, *H*2= {0, 1, 0, 0}, *H*3 = {0, 0, 1, 0}, *H*4= {0, 0, 0, 1}.

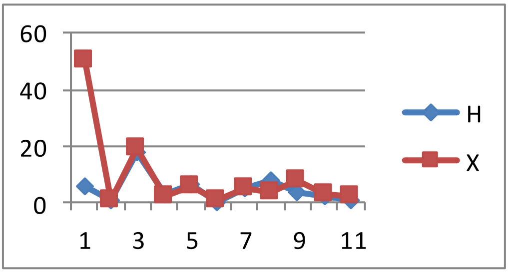

Figure 3. Proximity visualization of H & X from the Pn space.

Using polynomial matrices and get the

$$E = \begin{vmatrix} f0 & f1 \\ f2 & f3 \end{vmatrix} \rightarrow f0 * l0(t) + f1 * l0(t) + f2 * l2(t) + f3 * l3(t).$$

E cannot be shown in 3D, and therefore it is being substituted with a 2D line E1(t) [6]. Let's see how the graphic would look for the following values H:$\{54, 1\ , 18, 2.6, 6.4, 0.2,\ \ 4.7,\ \ 8, 3.3,\ \ 2\}$

X:$\{50, 1\ , 19, 2.4, 6, 1{,}5,\ \ 3.8,\ \ 8, 3,\ \ 2.5\}$

Figure 3 illustrates the result of the test model, maintaining relative proximity.

4 Conclusion

In this paper, we outline a model that enables effective visualization of multidimensional data. Next, we will apply the model to real-life data to evaluate its effectiveness for exploratory data analysis and data clustering.

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Romanian Spelling and Grammar Checking Systems

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Abstract

This article provides description about spelling and grammar checking system developed for detecting various grammatical errors in Romanian sentences. This system utilizes a full form lexicon for morphological analysis, and applies rule-based approaches for part-of-speech tagging and phrase chunking. The system can detect and suggest rectifications for a number of grammatical errors, resulting from the lack of agreement, order of words in various phrases.

Keywords: JSON data, JavaScript, POS, spelling system, grammar checking system.

1 Introduction

Romanian spelling and grammar checking system is a program that corrects a sentence at the word and syntactic level. The system rearranges necessary data depending on the sentence structure through semi-structured data. The system cannot correct 100% of what the user aims. The program is divided into two systems: Correct Spelling System and Correct Grammar System.

The Correct Spelling System checks each word from the sentence if the word exists in dictionary. If it does not exist in dictionary, the system finds similar words from the dictionary and offers them to the user.

The Correct Grammar System checks the form of the components of the sentence from the basis form of the sentence, and if it does not suite for processing, the system corrects the form depending on the subject of the sentence.

2 Check and correct spelling system

Check and correct spelling system uses JavaScript program language and JSON data. A user inputs a sentence and checks each word from Romanian dictionary. In addition, if the words do not exist in JSON data, the processing engine finds similar words and recommends the words to the user.

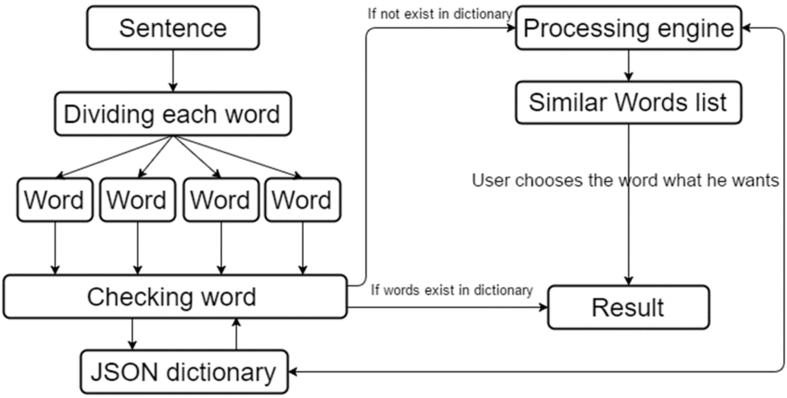


Figure 1. Architecture of correct spelling system.

At the first step it is necessary to make a Romanian dictionary in JSON data. Here is an example JSON data. It is made into an array:

{"words":["pisică","câine","tigru","păsări","pește","crocodil""]}.

The dictionary is formatted in JSON. Here is the source [1], which contains more than 100,000 Romanian words.

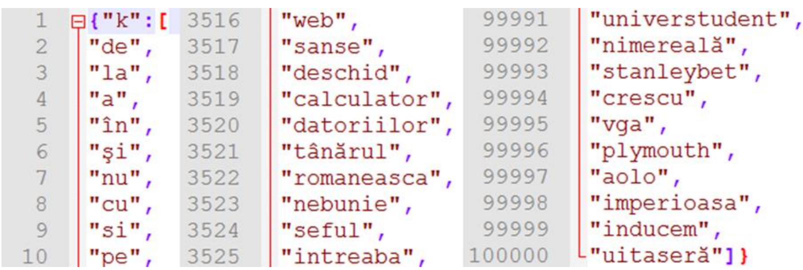


Figure 2. The dictionary formatted in JSON.

At the second step the system verifies if the word exists or not. If the word exists in JSON, the message "cuvîntul acesta este în dicționar" appears, else if the word doesn't exist in JSON, there appears the message "cuvîntul acesta nu este în dicționar".

At the third step, if the word doesn't exist, the system finds the corrected word with the help of processing engine. Processing engine is used by Edit Distance [2]. Edit Distance is a way of quantifying how two dissimilar strings (e.g., words) are sticked together by counting the minimum number of operations required to transform one string into the other. Edit Distance finds applications in natural language processing, where automatic spelling correction can determine candidate corrections for a misspelled word by selecting words from a dictionary that have a low distance to the word in question.

Processing engine should work like the following examples.

- calculatoi → calculator. The symbol "i" in the word changes to "r".
- calclator → calculator. The symbol "u" is added anto the word between the symbols "c" and "u".
- ccalculator → calculator. The symbol "c" is deleted from the word.

Here is the result calculated by Edit Distance.

Figure 3. Results calculated by Edit Distance for word "calculator".

Approximately 600 new words were generated by Edit Distance function.

3 Check and correct grammar system

Check and correct grammar system uses JavaScript program language and JSON data. A user inputs a sentence, the system checks each word of the sentence and associates it with the respective Part of Speech (POS) using the dictionary of Romanian POS. If the sentence is not correct from the Romanian grammar point of view, the program rearranges the words at syntactic level.

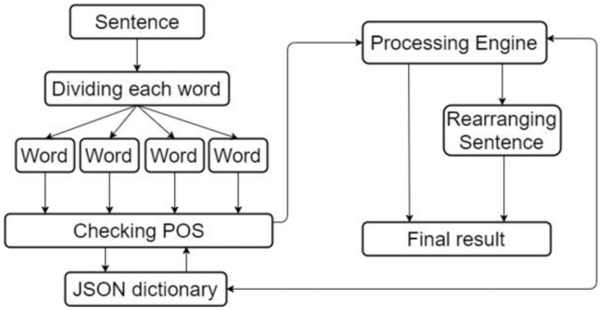


Figure 4. Architecture of correct grammar system.

The first step divides the input sentence word by word, and puts them into an array.

At the next step, it is necessary to make a POS of Romanian dictionary in JSON data. The MULTEXT-East [3] resources are a multilingual dataset for language engineering research and development. This dataset contains Bulgarian, Croatian, Czech, English, Estonian, Hungarian, Lithuanian, Macedonian, Persian, Polish, Russian, Romanian, Russian, Serbian, Slovak, Slovene, and Ukrainian languages.

The MULTEXT-East project adapted existing tools and standards to those languages.

The database with POS code type can be obtained from the site nlptools.info.uaic.ro [4]. This database has approximately 1.1 million words with POS code type included.

Example word "Calculator", code is "Ncmsrn":

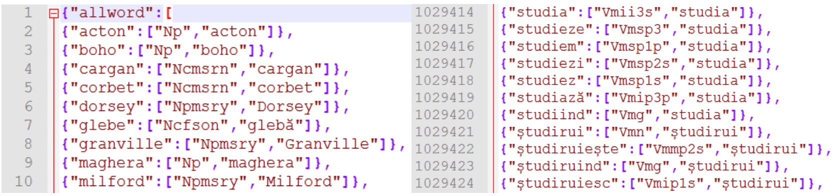
"N"- Noun, "c" - Common, "m" - Gender Masculine, "s" - Number Singular, "r" - Case Direct, "n" - Not Defined.

Therefore, JSON form will be like the following:

{ "WORD" :["POS Code" , "OriginalWord" , "InfiniteWord"] }

Here is an example of JSON data with POS code type.

{"allword":[
{"calculator":["Ncmsrn","calculator"]},
{"merge":["Vmip3s","merge"]},
{"repede":["Afpfsrn","repede"]},
{"studiat":["Afpmson","studiat","studia"]},
]}



```
{"allword":[
{"acton":["Np","acton"]},
{"boho":["Np","boho"]},
{"cargan":["Ncmsrn","cargan"]},
{"corbet":["Ncmsrn","corbet"]},
{"dorsey":["Npmsry","Dorsey"]},
{"glebe":["Ncfson","glebă"]},
{"granville":["Npmsry","Granville"]},
{"maghera":["Np","maghera"]},
{"milford":["Npmsry","Milford"]},
```

```
{"studia":["Vmii3s","studia"]},
{"studieze":["Vmsp3","studia"]},
{"studiem":["Vmsp1p","studia"]},
{"studiezi":["Vmsp2s","studia"]},
{"studiez":["Vmsp1s","studia"]},
{"studiază":["Vmip3p","studia"]},
{"studiind":["Vmg","studia"]},
{"ştudirui":["Vmn","ştudirui"]},
{"ştudiruieşte":["Vmmp2s","ştudirui"]},
{"ştudiruind":["Vmg","ştudirui"]},
{"ştudiruiesc":["Vmip1s","ştudirui"]},
```

Figure 5. POS of word in JSON

arrSen = ["calculator", "merge", "repede"];
posSen = ["Ncmsrn", "Vmip3s", "Afpfsrn"];

When POS code types were associated to each word, the sentence is corrected by the rules of Romanian grammar. For example, in the textbox the following sentence is input: "Calculatorul merg repede". The sentence is incorrect because the word "merg" is a Verb Form of the First Person type. Therefore, the system must find in array the word, which has the verb form of the Third Person type.

{"merg":["Vmsp1s","merge"]}
"merg" is parent key and "Vmsp1s" and "merge" are childern keys.
{"merge":["Vmip3s","merge"]}
"merge" is parent key and "Vmip3s" and "merge" are childern keys.

So the system can find correct verb by children keys, the original word of "merg" namely "merge" and the correct verb form of Third Person type "Vmip3s".

4 Conclusion

Romanian Spelling System and Grammar Checking System can check if input words exist in dictionary or no. In addition, they can transform incorrect forms of words to the correct ones. These systems find where incorrect verbs, verbal nouns, adjectives, articles, nouns, pronouns and some particles are. However, there are also things that this program cannot do. Another problem is that computer cannot understand the meaning of some words.

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Algebraic Methods in Automated Reasoning

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Abstract

We present succintly a case study on automatic generation of natural–style proofs in elementary analysis, by employing algorithms from computer algebra. In order to produce proofs which are similar to those realised by human mathematicians, we use a system similar to sequent calculus, in which the most difficult steps consist in finding the witness terms for the existential goals and the instatiation terms for the universal assumptions. We study how these can be found by using computer algebra algorithms, and what are the current limitations and perspectives of this approach.

Keywords: computer algebra, natural–style proving.

1 Introduction

The production of natural–style proofs (that is: proofs which are similar to the ones written by human mathematicians) may be of increasing importance in the future, because understanding proofs may become crucial in order to trust them, or to guide the difficult steps, or to use them in tutorial presentations. The *Theorema* system [1] aims at constructing such proofs in various areas of mathematics.

For proofs in elementary analysis, in which many notions are defined using complex formulae with alternating quantifiers, we developed the original strategy of *S-decomposition* [2], which is particularly suitable for treating such formulae. In such proofs, the tasks which are most difficult to automate consist in finding the witness terms for the existential goals and the instatiation terms for the universal assumptions. Our case study in *Theorema* demonstrates how these tasks can be partially solved by using cylindrical algebraic decomposition and quantifier elimination [3]. Although in linear cases this approach is mostly successful, in problems of higher degree it often fails. We investigate various methods to improve the performance in these cases.

2 Natural–Style Proving

In the *Theorema* system we aim at producing proofs which are similar to those realised by humans[1]. For instance, let us consider the proof of the statement: *"The sum of two convergent sequences is convergent"*. The convergence of a sequence f (function from naturals to reals) is defined by the following formula with alternating quantifiers (ϵ is real, m, n are naturals):

$$\underset{a}{\exists}\ \underset{\epsilon>0}{\forall}\ \underset{m}{\exists}\ \underset{n\geq m}{\forall}\ |f(n)-a|<\epsilon$$

The proof shows that the instance of this formula for $f_1 + f_2$ (the *goal*) is implied by the two instances of the same formula for f_1 and for f_2 the (*assumptions*).

The natural–style proof proceeds by eliminating in parallel the same quantifiers from these three formulae, as described in [2]: In the existential assumptions, the quantified variable is transformed into the so called "such a" Skolem constant, and after that the existential goal is proved by using the appropriate "witness term". In the universal goal, the variable is transformed into the so called "arbitrary but fixed" Skolem constant, and after that the universal assumptions are instantiated with the appropriate terms. A special feature of our approach is to treat separately the condition associated to the quantified variable (in the formula above: $\epsilon > 0$ and $n \geq m$), which generate separate independent goals.

[1] This is not the same as *natural–deduction*.

3 Using Algebraic Methods

While the Skolemization steps mentioned above have a trivial implementation, the construction of the witness terms and of the instantiation terms is quite difficult to perform automatically, because the necessary information becomes available only later in the proof. We experimented the use of algebraic techniques for finding these terms, following a method presented in [4]. In the example above, the successive steps of the proof are essentially equivalent to a prenex decomposition of the whole original implication, and formulae obtained are:

$$\underset{m,n}{\forall}\ \underset{p}{\exists}\ \underset{q}{\forall}\ (q \geq p \implies q \geq m \wedge q \geq n)$$

$$\underset{a_1,a_2}{\forall}\ \underset{a}{\exists}\ \underset{\epsilon}{\forall}\ \underset{\epsilon_1,\epsilon_2}{\exists}\ \underset{x_1,x_2}{\forall}\ (|x_1-a_1| < \epsilon_1 \wedge |x_2-a_2| < \epsilon_2 \implies |(x_1+x_2)-a| < \epsilon)$$

For proving the first formula we can use CAD–based quantifier elimination (QE), and the answer is **true**, but this does not reveal a natural–style proof. If we use QE on the same formula without $\underset{m,n}{\forall}\ \underset{p}{\exists}$, then we obtain a relation between m, n, p which allows to infer the expression for p (will be the maximum of m and n) by adequate postprocessing. For proving the second formula, one can apply QE/CAD first on the formula without $\underset{a_1,a_2}{\forall}\ \underset{a}{\exists}$, which returns $a = a_1 + a_2$. Then one substitutes a and eliminates further the quantifiers $\underset{\epsilon}{\forall}\ \underset{\epsilon_1,\epsilon_2}{\exists}$, on which QE/CAD returns $\epsilon_1 + \epsilon_2 \leq \epsilon$, which allows to infer appropriate witnesses for ϵ_1 and ϵ_2, namely $\epsilon/2$.

The above approach is not very efficient, because it needs a repeated CAD for each formula. Therefore we are investigating possible adaptations of the algorithm which can extract all the necessary information in one pass. Moreover, while the algorithm works relatively fast for expressions of degree one (as above), it is overcoming the system resources for expressions of higher degree (for instance, when we try to do the analoguous proof for *product* instead of *sum*). For overcoming this problem we are operating various simplifications of the original formulae, which need less computation, but still are able to reveal the same desired terms.

4 Conclusion

The use of algebraic algorithms for producing specific terms in natural–style proofs is successful at least in simple cases, however for more complex problems it becomes unproductive. Performing various case studies in elementary analysis appears to hold the promise of finding more efficient and effective versions of the algorithms, which will be able to solve more complex problems.

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Preparation of article-level metadata with bibliography lists

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Abstract

In this paper we present the problem which arises in connection with article-level metadata preparation for international databases, especially when these metadata are to contain bibliography lists. This problem leads to the necessity of automatized approach to the process of bibliography making in this framework.

Keywords: indexing, metadata, automatization process, impact factor.

1 Introduction

Nowadays, scientific community becomes more and more concerned with not only presenting its research results, but getting greater indexing recognition as well. So, researchers seek to publish their results in journals with high impact factor, and scientific journals in their turn tend to be present in the most prestigious international databases such as SCOPUS, Thomson Reuters, INSPEC, zbMATH, DOAJ, etc.

Each article is analyzed, followed and processed (indexed) by specific databases. However, its true usefulness for scientists will be determined by the citations received by its articles. Each article is represented by bibliographic information and citation information which points to original articles. [1] This information is used by bibliographic and citation databases in their tools that search and follow the articles.

The databases are now updating the lists of journals that they follow up and ask respective editorial boards to provide updated information about the journal, at the same time they are asking for metadata about journals and articles published by them. In different databases there are different requirements and different format of metadata presentation.

In this paper we describe the importance of a tool that would automate the process of obtaining the metadata. A system was developed and used already by the Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova. It helps us to solve the problem of bibliography automatizing when preparing article-level metadata – the process which is necessary if the journal tends to obtain a higher impact factor. The goals of this project are to reduce the preparation time for metadata and to obtain more accurate and complete information about an article.

2 Article-level metadata

The metadata created for journal articles by authors and publishers allow them to document key elements relating to their works. The most concrete definition about metadata says that it is data that provides information about other data. Metadata consists from information that describes the content, quality, history, availability and other characteristics of the data. In other words, metadata is a container element for information concerning the article that identifies or describes the article. [2]

As to the field of journals and books publishing, we need to consider such metadata as *articles metadata*. Such type of metadata may contain bibliographic data (authorship, article title, copyright year, and publication date), descriptive material (keywords and abstracts), and any numbers identifying an article. Some journals distinguish three types of article metadata: about the specific article, about the journal, and about the issue of the journal containing the article. [3]

When creating metadata for a journal article, the author or the publisher identifies its basic elements: the author, title, journal name, volume number, issue and creates a Document Type Definition (DTD) consisting of those elements. A DTD is a structured, tagged representation of an article. This DTD usually is created according to the requirements of the respective database or community. Using a DTD and data files, information regarding your articles can easily and accurately be communicated to other members of scientific community, publishers, or online service providers.

3 Bibliography in article-level metadata

Initially, the list of references in scientific articles was intended for the reason that authors could be based on the already published. Using this list, if necessary, readers can find and read more relevant information concerning the problem described in the article. To facilitate the search for the publications cited, the information must be as much accurate and complete as possible.

For greater indexing recognition, scientific journals are interested to be present in the most prestigious international databases in the respective domain, for example, for Computer Science these are SCOPUS, Thomson Reuters, INSPEC, zbMATH, DOAJ, etc. Each database elaborates its own format for article-level and journal-level metadata.

zbMATH is one of the biggest databases that offers access to bibliographic data, reviews and abstracts both in pure and applied mathematics areas. They have been developing a new specific XML input format – zbJATS based on the Journal Article Tag suite by NBCI.

The process of bibliography automating becomes more difficult if the information provided by authors is incomplete or does not correspond to the journal requirements. Therefore, we kindly ask authors to provide complete and accurate information about sources. This information has to be written in LaTeX in accordance with the requirements of Computer Science Journal of Moldova (CSJM) based on IEEE2015 standard.

An article can have many bibliographic references, and writing all this information by hand, in XML format, requires a lot of time. That's why we consider that it is necessary to develop a system that will automate this process according to international standards and will guarantee good results in a record time.

4 Practical application and method of algorithm realization

Using a system whose purpose is to automate the process of obtaining the metadata will save time, money and also will offer complete information in a record time.

As input data we use articles written in LaTeX-e, according to the requirements of CSJM. Below we have an example of how a journal reference written in LaTeX should be provided as a result in XML format:

```
<ref>
  <element-citation publication-type="journal" publication-format="print">
        <name>
              <surname>Pustejovsky</surname>
              <given-names>J.</given-names>
        </name>
        <article-title>The Generative Lexicon</article-title>
        <source>Computational Linguistics</source>
        <volume>17</volume>
        <issue>4</issue>
        <fpage>409</fpage>
        <lpage>441</lpage>
        <year iso-8601-date="1991">1991</year>
  </element-citation>
</ref>
```

The system recognizes the bibliographic references of articles using the regular expressions. For every specific element of an item, we construct search patterns using different characters. For example, the following search pattern searches in the reference only the author names: `"/\\\bibitem\{[0-9a-zA-Z, _-]+\}(.*?)\\\\textit/i"`.

The developed system has processed already 4 issues, which means 32 papers of CSJM. Our experience has shown that approximately 85% of references are processed properly by the system. Those remained erroneous are because of the authors' carelessness or some nonstandard item which was not foreseen in the system.

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Variety of stroke prediction models, risk factors and underlying data sets

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Abstract

This work is devoted to the analyses of variety of stroke prediction models, risk factors and underlying data sets taken into account. The aim is to define the most appropriate ones for stroke prediction model construction in the conditions of Moldova clinics and population.

Keywords: stroke prediction models, risk factors, underlying data set, NIHSS score, Bartel scale.

1 Introduction

At present, there is a great variety of stroke prediction models. For a long time, stroke remains at one of the first places in the world as a cause of death and disability of the population. Therefore, the concern of the entire population and medical community, in particular, about the prevention of this dangerous disease, is understandable. And also, the interest of neuropathologists is clear when they want to be able to predict the occurrence of this disease, to identify risk groups among the population, to prescribe preventive measures and treatment that could prevent the disease or reduce its severity. So, it is understandable the large number of prognostic models, in which different goals are set, various situations of stroke, as well as various risk factors, are taken into account. Different data sets are available for models construction as well.

Moreover, the practice of using these models has shown that models are individual in their application both geographically and in time. Models developed in one country, for example, in the US, may be not suitable for use, for example, in Germany or Moldova, and even within the same country, but in different clinics or for another population subset. Models developed 20 years ago cannot be used (or will yield results with much less accuracy) today. This is because with the course of time the living conditions of the population change, the susceptibility of the organism changes, and the risk factors change.

In this work we describe the attempt to analyse stroke prediction models, the picked out risk factors and used underlying data sets. The aim is to define the most appropriate ones for model construction in the conditions of Moldova clinics and population.

2 Stroke prediction models and tools used

In the framework of the project “Mathematical modeling of risk factors and clustering of patients for preventive management of stroke” the group of researchers from the Institute of Mathematics and Computer Science is going to create mathematical model which will be able to differentiate patients which are potentially at the risk of having stroke. Thereby, people that are at high risk will get adequate preventive treatment.

In the intensive care units for neurological patients and neurorehabilitation of the Federal State Institution (FSI), “Medical and Rehabilitation Center of the Ministry of Health and Social Development of Russia”, the research was conducted and models were developed for predicting functional outcomes after a severe and extremely severe stroke [1]. The purpose of these models is to predict the likely scenario of the development of the disease on the basis of a set of initial characteristics. If, as a result of the curative effect, the scenario improves, then therapy is recognized as effective, and vice versa.

There was developed a model of assessment prediction on the Bartel scale in 1, 3, 6 and 12 months after stroke, which includes initial assessment on the Bartel scale and National Institutes of Health Stroke Scale (NIHSS). For patients with brain infarction in the medial cerebral artery basin it was expedient to include additionally into the model an assessment on the ASRECTS scale. When predicting unfavorable outcome (5-6 numbers on Rankin scale) the most significant is an assessment according to the Glasgow Coma Scale (GCS) on admission.

The complex model gives the more accurate forecast: Complex_index = 7.14 – 0.07 NIHSS – 0.548 GCS – 2.91 Bartel (0) – 0.005 × (transverse dislocation, mm) – 1.03 × (axial dislocation, mm).

These models contain only generally available indicators, so their introduction into clinical practice will not require additional financial costs. At the same time, an early assessment of the prognosis will allow the development of individual rehabilitation programs and monitor their effectiveness at different recovery times after a stroke.

Another 3 models were developed using the data on 538 consecutive acute ischaemic and haemorrhagic stroke patients enrolled in a Stroke Outcome Study [2]. These models use different sets of variables: from simple clinical ones to the set with added more complex clinical variables and information from the first computed tomography (CT) scan. For Model I there were taken age, pre-stroke independence, arm power and a stroke severity score. It appeared that it didn't perform better than Model II with age, pre-stroke independence, normal verbal component of the Glasgow coma score, arm power and being able to walk without assistance. Model III with simple clinical variables and two radiological variables was not statistically superior to model II. Models were developed using multivariate logistic regression analysis. The authors consider that outcome prediction was not significantly improved with CT-derived radiological variables or more complex clinical variables [2].

There is some other direction for prediction models for stroke: mortality prediction after hospital admission for ischemic stroke [3]. Usually, cases of hemorrhagic stroke or transient ischemic attack were not included. Individual risk of mortality of a patient at admission is a valuable criterion for determining adequate clinical care and identification patients at high risk for poor outcomes who require more intensive resources. One of the examples of such models is World Wide Web–enabled bedside tool for risk stratification at the time of presentation for patients hospitalized with acute ischemic stroke [3]. Data from the Get With the Guidelines–Stroke (GWTG-Stroke) database were used. For this study there were used cases of ischemic stroke in 274 988 investigated patients registered in 1036 hospitals during almost 7 years. Cases were randomly divided into a derivation (60%) and validation (40%) set. Also it is important to be noted that another model was derived and validated for

109 187 patients (from those 274 988) with a NIHSS score recorded. It was done because NIHSS score was not normally documented for all the patients, but it was interesting to test the hypothesis that NIHSS score would be a strong determinant of mortality. To determine the independent mortality predictors and to assign point scores for a prediction model, logistic regression was used. Model distinction was measured by calculating the C statistic from the validation sample. In-hospital mortality was 5.5% overall and 5.2% in the subset with recorded NIHSS score. Risk factors associated with in-hospital mortality were age, arrival mode, history of atrial fibrillation, previous stroke, previous myocardial infarction, carotid stenosis, diabetes mellitus, peripheral vascular disease, hypertension, history of dyslipidemia, current smoking, and weekend or night admission. The C statistic was 0.72 in the overall validation set, 0.85 in the model with NIHSS score and 0.83 in the model with NIHSS score alone. One of the authors' conclusion is that "The NIHSS score provides substantial incremental information on a patient's short-term mortality risk and is the strongest predictor of mortality" [3].

It is obvious that comparison of different models is a very complex task and in many cases is not proper. But from the other hand, it is interesting to take in consideration the previous experience. To easier appreciate the diversity of models' parameters (input and output) from different points of view and to choose those interesting for the moment for comparison or discussion, the summary table is being created. The models in it are sorted by year of publication of the corresponding paper describing the model. The table includes characteristics of underlying datasets, methods used and models' output parameters as well. Table 1 shows its fragment.

At present time almost 40 research papers describing stroke prediction models are being considered by us. Some of them present the corresponding models which can be used by the readers, some give access to dataset used. All this helps in understanding the significance of the parameters used and construction of new models.

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Table 1. Fragment of the summary table with 41 input parameters of 15 analysed stroke prediction models

Model[1]	[4]	[5]	[6]	[7]	[3]1	[3]2	[1]1	[1]2	[1]3	[1]4	[8]1	[8]2	[8]3	[8]4	[9]
Input parameter[2]															
Age	+	+	+	+	+		-	-			+	+	+	+	+
Gender	+		+	+			-	-							+
Axial dislocation							-	-		+					
Transverse dislocation							-	-		+					
Dyslipidemia.					+										+
Abdominal Circumference															+
HLV on Echocardiogram															+
HLV EKG Sokolov-Lyon															+
Glomerular Filtr. Rate															+
LV Wall Thickness															+
ECG-LVH	+	+	+	+											
Postprandial Glycaemia															+
A Jeun Glycaemia															+
Previous Conditions[3]	+		+	+	+										+
Glasgow Coma Scale							-	-	+	+					
NIHSS						+	+	+		+	-	-	+	+	
Bartel scale							+	+		+					
systolic blood pressure	+	+	+	+											
diastolic blood pressure		+													
use of anti-hypertensive ther.	+			-											
hypertension					+										
diabetes mellitus	+	+	+	+	+										
impaired fasting glucose			+												
atrial fibrillation by ECG	+		+	+	+										
creatinine level			+	+											
time to walk 15 feet			+	+											
Carotid stenosis					+										
Current smoker		+	-		+										
Peripheral vascular disease		+	-		+										
Coronary heart disease		+	-												
% HDL cholesterol		+	-												
Living alone											+	+	-	-	
Independent pre-stroke											+	+	-	-	
Normal GCS verbal score											+	+	-	-	
Able to life both arms											+	+	-	-	
Able to walk unaided											+	+	-	-	
arrival mode					+										
previous stroke					+										
weekend or night admission					+										

[1] Source in references list and /model's number; [2] - insignificant variables, + significant variables;
[3] Previous Conditions: stenocardia, cardiac infarction, coronary revasculization

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Treebank Annotator for multiple formats and conventions

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Abstract

The UAIC-RoDepTreebank becomes an important balanced corpus, with increasing dimensions, which is intended to be used for multiple applications. For this purpose, it will be available in several formats, classical syntactic, semantic, Universal Dependencies and PROIEL. Dependency trees are in XML format, and for viewing it, we use frameworks that allow manual annotation or automatic annotation correction. As the interface used so far only allows working with the classic syntactic format, we present here a new, multifunctional interface that allows the input of any tree format.

Keywords: natural language processing, dependency treebank, manual annotation, automate supervised annotation.

1 Introduction

Corpora are very important resources for Natural Language Processing, in the absence of which there is no possibility of training the programs and of making searches. The degree of computerization of a language depends on the existence of large corpora with many types of consistently annotated information.

UAIC-RoDepTreebank **[4, 5]** has complete morphological analysis **[9]** and syntactic classic annotation, in Dependency Grammar **[6]**, entirely supervised. The classical syntactic format has now 19,825 sentences and 389,357 tokens, punctuation included. The specificity of our corpus is the tendency to cover all the styles of the language, being more interested in the non-standard language. We already have 2,575 phrases on chat communication and 6,882 phrases in old Romanian, written from the seventeenth to the nineteenth century, although it is difficult to create and train tools to process these types of texts.

In order to make the corpus accessible for multiple applications and as many users as possible, we decided to transpose it into an international format, Universal Dependencies, (UD), maintained by a group that brings together over 30 treebanks **[8]**. Currently, the UD format of our treebank has 4,600 sentences and 101,568 tokens, and other sub-corpora are in the course of transformation.

Another purpose of our work is to add new complex annotations to the corpus, so we started creating another format, in which the syntactic trees are transformed into semantic dependency trees. This format has now 4,405 sentences with 72,607 tokens, and other sub-corpora are in course of transformation.

As the trees are in XML format, (or CONLLU for the UD format) **[1]**, annotators and users have the necessity to visualize, and eventually modify them using a framework. The one used so far has been created according to the classic format of the treebank and only allows uploading sub-corpora in this format.

We present here a new multifunctional and flexible framework used currently for dealing with the new (UD and semantic) formats of our treebank. The tool has been created as a dissertation project **[2]**, being able to perform all the new tasks of the treebank that the old annotator could not fulfill. Each function has been experienced by the user up to its perfect operation, and the drop-down lists have been aligned with all the new changes of the tag lists.

2 Related Work

We present shortly four other similar programs. The old *Treeannotator* **[7]** is an application built on the Java platform. It allows loading only a particular format. If there are inconsistencies in the corpus loaded, the tool brings the corpus to its standard form. It restores the numbering of words in sentences, puts the items of XML in the same order, checks the validity of the XML, and shows where is the mistake that makes the XML invalid, marks the graphs that are complete trees in the list. It can draw the sentence graph in a linear manner or in form of trees.

The proposed interface on the UD site is *Dependency Grammar Annotator* (DGA) **[10]**. The format that can be loaded is UD, and the graph is linear.

Easy Tree is an application running in the browser, created within the CQL summer program, edition 2015 **[3]**. It allows the editing of small sentences. The display is in tree form.

Grammar Scope is an application on the Stanford Parser \ Stanford CoreNLP platform **[11]**. It offers very advanced functions of editing, creating and parsing linguistic resources, syntactically and semantically annotated, in a different format than ours.

3 Treebank Annotator Settings

Although *Grammar Scope* is a very good tool, it cannot be used by us because it was made for different format. Although these tools are language-independent, some specificities of a language make the authors to choose specific annotation conventions. We can see that each corpus creates specific automatic and manual annotation tools that are presented on its site.

The new annotator of the UAIC-RoDepTb corpus performs all functions of the Treeannotator described above, and adds many more. It works with the folder called ”Configurations” that contains working files with formatting features with the lists of all possible values. The interface has a drop down list for each feature. We can add a new file in the folder ”Configurations”, to introduce another format, or a new feature in a configuration, or a new value allowed in the list of a feature, or a new list of allowed values.

In order to open a sub-corpus (which has around 1,000 sentences) it is necessary to choose first a configuration, i.e. to specify in which format the corpus to be opened is.

Concerning the XML and CONLLU format, the application allows uploading of one of them and saving in the other, i.e. it can function as convertor. The application also allows uploading simultaneously more sentences in different formats and comparing two of them. (see Fig.1).

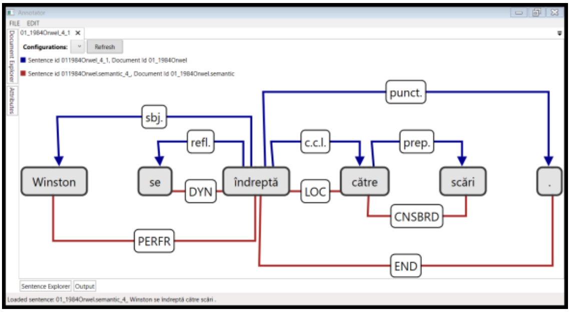

Figure 1. Comparison between the syntactic and semantic format. (En: Winston made for the stairs.)

4 Other facilities offered by Treebank Annotator

There are three possibilities of graphs visualization: linear, tree or oblique. The linear order is preferred because it is the only one that allows the tracing of dependencies simultaneously with the order of words in the text. It would be desirable for the tree view to allow the order of the words to be traced, as is the case in the old Treeannotator.

The new application allows uploading of a treebank or creating a new dependency tree, introducing a text to be manually annotated. The tool creates an XML for the inserted text, with all the features of the format chosen by default, and the user selects a value for each feature, from its drop-down list.

The tool allows not only changing the edges and the tags of all features, but also adding or removing words. It is an important function, because some multi word expressions (MWEs) are not correctly interpreted by the tokenizer. A group of words may be misinterpreted as MWE, so the words need to be separated, or multiple words have unitary meaning, forming a MWE, and need to be united.

When we are adding or removing a graph node, the application automatically changes both word and head ID-s, so that the rest of the graph does not change. The tool also allows changing the word order in the sentence, without changing the dependency structure. When a word is added, it is placed at the end of the sentence and then, by changing the order of words, it is brought to the desired location.

When a word is selected, all its features existent in XML appear in the ”Attributes” field, each with its dropdown list. The existence of dropdown lists allows selecting a tag by typing only the first letters and eliminates the possibility of mistyping. The program does not allow writing a tag that is not in the chosen Configuration list.

5 Conclusion and future work

Specific tools have to be built for each treebank corpus, respecting the distinctive rules and conventions, even if they are intended to be language independent. In the paper, we have presented a new tool created for the needs of UAIC-RoDepTb.

Following the model of *Grammar Scope* cited above, a syntactic or/and semantic parser will be integrated in the Treebank Annotator. Currently, the tool is used with success for the supervision of automatic transformation of the syntactic classic format in the UD or in the semantic one. In the future, tools for the automatic transposition of the classical syntactic format into UD and into semantic ones will be integrated as convertors in the Treebank Annotator.

Another project of our NLP group is to create a Pattern Dictionary of Romanian Verbs (PDRoV), which will be accessible online. It will include contemporary, archaic, regional, familiar verbs. The Patterns are structures of mandatory or facultative syntactic dependencies, with their semantic possible realizations.

For each verb described, examples of all the patterns will be included in the dependency treebank, in all of its three formats. The corpus in the three formats will form a database linked to the PDRoV site.

The Treebank Annotator will be a very important component of this project, permitting the user to visualize the tree form of each example for each pattern described, in the format chosen, classical, UD, or semantic, displayed in linear, tree or diagonal view.

This interface is important because it respects all the conventions of annotation of our treebank and also permits us to make changes in its Configurations, to meet new requirements, format changes, or new formats introduced to synchronize with similar international projects.

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News on Twitter

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Abstract

In this paper, we present an application built by us with aim to provide users the possibility to explore topics from Twitter and find out people's opinion even if it is a positive one or a negative one. Using algorithms that calculate distance between two strings, similar tweets will be removed.

Keywords: Twitter, strings similarity, sentiment analysis.

1 Introduction

Every day, millions of people use Twitter to create, discover and share ideas with other. From local stores to big brands, and from brick-and-mortar to Internet-based or service sector, people are finding great value in the connections they make with businesses on Twitter[1]. There are many great business uses for Twitter, like sending out news briefs or advertising the latest job opening.

Similar applications: IceRocket[2] is generally for blog searches, but it offers the possibility to search news on Twitter. Twitter search will return most recent tweets that relate to your query. If the query is also a user, it will show a fact box about the user, along with tweets by that user. **Twitonomy**[3] is an online platform and in order to use it you have to connect with your Twitter account. The user has the opportunity to monitor his account or any other Twitter user, along with lists and any keyword search he wants to watch.

[1]http://askaaronlee.com/10-reasons-why-your-business-should-use-twitter/
[2]http://www.icerocket.com/
[3]http://www.twitonomy.com/?gclid=CO-gi_fZ8dECFQuMGQodbgYCkQ

2 Application description

The developed application aims to offer users the possibility to follow news on Twitter, without having to read duplicate topics. In order to remove similar news, it was used a similarity algorithm which calculates distance between two tweets. In the first phase, it was made an analysis, in terms of time and accuracy, of the next four similarity distance algorithms: Levenshtein, Needleman-Wunsch, Jaro-Winkler and Smith-Waterman. Upon review of the analysis result, Smith-Waterman turned out to be more competent to find similarities.

When the user performs a search in application, it is done a request to Twitter API in order to retrieve the latest and the most popular tweets. The search result will be divided into subcategories. Each category will be extracted from tweets and can be: *a location, a person, an organization* or *a date*. Also tweets will be analyzed from the sentiment point of view (*positive, negative* or *neutral*).

Similarity algorithms: In computer science and statistics, the **JaroWinkler distance** is a measure of similarity between two strings [1, 2]. It is a variant of the **Jaro distance** metric a type of string edit distance, and was developed in the area of record linkage (duplicate detection). The **Levenshtein distance** is a string metric for measuring the difference between two sequences [3]. Informally, the Levenshtein distance between two words is the minimum number of single-character edits (i.e. insertions, deletions or substitutions) required to change one word into the other. The **Needleman-Wunsch algorithm** is an algorithm used in bioinformatics to align protein or nucleotide sequences [4]. It was one of the first applications of dynamic programming to compare sequences. The **Smith-Waterman algorithm** is a dynamic programming method for determining similarity between nucleotide or protein sequences [5]. The Smith-Waterman algorithm is build on the idea of comparing segments of all possible lengths between two sequences to identify the best local alignment.

Named entity recognizer module: This module deals with extracting information, localizes and classifies named entities in tweets into pre-defined categories such as the *names of persons, organizations, locations, expressions of times*, etc. For all these operations it was used "Stanford Named Entity Recognizer (NER)" library[4].

Tweets classifier module: Formerly known as Twitter Sentiment, Sentiment140[5] is a service that lets users discover the current sentiment around a brand, product or topic on Twitter. Sentiment140 uses classifiers based on machine learning algorithms and allows users to see the classification of individual tweets.

Best similarity algorithm: In the first phase of application development was aimed to select the best similarity algorithm. The four algorithms introduced above were applied on a set of 2,000 tweets. For detecting which tweets from the entire source data are similar, each tweet was compared with all that follow it. In the Table 1 it can be observed how long did the execution of each algorithm last. In order to improve the execution time, it was used a caching mechanism from Microsoft and we remove the stop words from tweets.

Table 1. Algorithms execution time

	Without cache	With cache	With cache after removing stop words
$Jaro - Winkler$	06:05.35	03:50.11	01:46.06
$Levenshtein$	22:35.37	12:10.08	11:26.82
$Needleman{-}Wunsch$	51:11.47	34:21.63	21:22.49
$Smith - Waterman$	39:46.32	35:33.20	23:51.46

Three of the algorithms find 505 similar tweets, while the Smith-Waterman finds 569 similar tweets (a tweet and a retweet are the same). In the end, we decided to use Jaro-Winkler algorithm due to its fast running time. In the application interface we have only one from the similar tweets and the tweets that were classified as positive have a green background and the negative ones have a red background.

[4]http://nlp.stanford.edu/

[5]http://help.sentiment140.com/api

3 Conclusion

Social networks can provide at a given moment a lot of information, especially in the moments when there are special natural phenomena, or when presidential elections take place. In such cases, the amount of data resulting from these networks are increasing exponentially, and there appears the need to apply advanced techniques like in this paper: to remove duplicates and to highlight sentiments from tweets.

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Development of the Decision Support System for Test Baking

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Abstract

The article describes the process of creating a computer decision support system for the quality of bakery products on the ground of characteristics of laboratory baking. Here is shown that large amounts of data can not be collected for statistical analysis by classical methods. The review of knowledge analysis tools used in the development was made and examples of product rules are given. The developed system can be useful for the operative regulation of the bakery products quality at the stage of regulating the formulation.

Keywords: decision support system, baking technology, mathematical modeling.

1 Introduction

At the stage of launching new products bakery production uses as a rule the method of trial laboratory baking, on the basis of which physico-chemical indicators are set for a specific name. For this purpose a portioning technique assuming a separate test batch for each lot of products. In order to obtain reproducible results, a recipe for 100 kg of flour is used, which makes it possible to establish new products with similar characteristics. At the same time, if the output measures do not comply with the standards, a repeated test batches is performed with the corrected recipe or with the corrected parameters in production process As a rule such work is conducted on the basis of knowledge and experience of the enterprise technologists. The research task is the development of an integrated system that allows to give recommendations on the operative regulation of the baking process based on the results of several test batches.

2 Subject field analysis

The quality evaluation of the finished product and providing recommendations for the rapid regulation of the formulation and process for the test lot is a complex analytical procedure, since it is necessary to evaluate various indicators, including: the volume of the baked product, humidity, acidity, porosity, appearance, crumb condition, and taste qualities. The main factors influencing the quality formation of bakery products are the type, humidity and ash content of the flour; quantity and quality of gluten, baking time, baking temperature, number of different ingredients.

Since it is not economically feasible to conduct a large number of production cycles, and the parameters of all products in the same test batches are identical, which is typical for any group process, then the data on which the solution is to be decided are both a small sample and a super-saturated plan.

Using a large number of samples from each batch is also impossible, as at the proper organization of the production process, the main characteristics of the batch are homogeneous and have practically no variations in the controlled parameters, which is reflected in the laboratory acts. Moreover, the usage of experimental data, which in fact is not statistically independent, will lead to the so-called problem of "imaginary repetitions" or pseudo-replication. It means that at regression model based on a such data set, estimating the mean square deviation of observed values from model predicted values will tend to exceed the true variance of observation errors. Such a reassessment will reduce the power of statistical tests and will make the hypothesis test of the model coefficients significance conservative, which can lead to incorrect interpretation of the results [1].

3 Design of the decision support system

For the solution of an objective it was decided to develop the decision-making support system using as tools: the technique of increase in the table of the researched data based on a method of pointed distributions [2], the ordinary least squares method, Dixon Q-test [3] for detection of the abnormal measurements and also a set of production rules for the rank variables analysis. At the stage of analyzing the available information, it was decided to build separate mathematical models for each of the output measures, which will form a system of equations that most fully describes the studied production process. In addition to knowledge of the quantitative parameters that are obtained by modeling, it is also necessary to add knowledge obtained from normative and technical documentation to the developing DSS vocabulary of terms, as well as knowledge from experts in the given subject area. On the basis of these knowledge was formulated a set of rules that relates individual characteristics of the technological process and recipes, allows to explain the occurrence of certain phenomena, and also to predict the behavior of the studied parameters.

Most of the rules that are included in the knowledge base are productive. Thus, the left side of the rule is a certain set of preconditions, and the right side contains an action that can be drawn from them. Such a rule comes into action if the preconditions associated with each other by a set of Boolean operations are fulfilled. In this case, each precondition is a certain value of one of the investigated parameters, which is expressed quantitatively.

Thus, in the currently under development DSS, the prediction of all quantitative parameters is based on a mathematical model constructed in accordance with the developed methods and algorithms. However, part of the investigated parameters are analyzed only organoleptically and are scoring, which makes it impossible to apply the above-described mathematical-statistical device to them. To manage the variations of such data, production rules are used that analyze both quantitative and scoring indicators. To construct mathematical models, linear regression analysis is used, which takes into account not only the influence of each individual variable factor, but all their paired interactions. If any of the input parameters has not changed during the whole production cycle, then it is not included in the consideration, since its variations can not be analyzed and evaluated. Analysis for the presence of outliers is carried out for all input and output parameters, if there is a suspicion of detecting such a value, the program displays a warning for the decision maker, with a recommendation to check the entered data, and this message does not affect the rest of the analysis procedure, which is performed in accordance with the algorithm.

4 Conclusion

Summarizing the above-described, it should be noted that the computer-based problem-oriented DSS will allow the process engineer to make more informed decisions about the operational regulation of the recipe and process for test baking of baked goods.

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Disaster response by simulation using GSPNs

Inga Titchiev

Abstract

The aim of this research is to provide potential solutions to respond in case of disaster and assist in the decision-making process by using Petri nets model [1, 2]. In order to provide these solutions logical and temporal dependencies have to be considered. For modeling there will be discussed information about the influencing factors and endangered objects in order of adequate response to the event. In these purposes quantitative and qualitative analysis will be done.

Keywords: petri nets, qualitative analysis, quantitative analysis, disaster.

1 Introduction

The Petri nets formalism [3] allows for the intuitive graphical representations of the modeled systems, as well as the analysis of the dynamic properties. One of the most important things is as a model system to work correctly, this is mainly determined by *qualitative* (or behavioral) properties. Another important aspect is to make sure that the system meets certain related performance characteristics (or *quantitative* properties). Petri nets allow checking the correctness of the modeled system at design phase.

2 Generalized Stochastic Petri nets

To perform quantitative and qualitative analysis, Generalized Stochastic PNs (GSPNs) [4] will be used. GSPNs have been extended from the ordinary PNs and contain the same basic sets such as:

- P the set of places;
- T the set of transitions, $P \bigcap T = \emptyset$;
- I, O, H are input, output and inhibition functions: $T \to N (N = P \bigcup T)$.

They were extended by using of two types of transitions: *immediate transitions* which are produced immediately and *stochastic transitions* which are produced complying with an exponential distribution function. Weighting function $W : T \to R$, was extended in the following way:

- for a timed transition (represented by a hollow rectangle), $w(t)$ is a rate (possibly marking dependent) of a negative exponential distribution specifying the firing delay;
- for an immediate transition (represented by a filled rectangle), $w(t)$ is a firing weight (possibly marking dependent).

The priority function $\Pi : T \to N$ associates the lower priorities to timed transitions and higher priorities to immediate transitions. The selection of which transition will fire is based on the priorities and weights. First, the set of transitions with the highest priority is found and if it contains more than one enabled transition, the selection is based on the rates or weights of the transitions. The initial marking $M_0 : P \to N$ determines the initial state of the modelled system. When a new marking is reached, if only timed transitions are enabled, this marking is called *tangible*; if at least one immediate transition is enabled, the marking is called *vanishing*.

3 Qualitative and quantitative properties

Qualitative properties of a Petri [4] nets are related to such properties as deadlock (no total deadlock), liveness (no partial deadlock), boundedness (on each place the number of tokens cannot grow in an unlimited way) and home state (markings that can always be reached from any reachable state). Reachability graph can be used to determine the boundedness. Also for checking the deadlock, liveness and bounddedness invariants analysis can be used.

Using qualitative properties and establishing additional restrictions quantitative properties can be computed. Average number of tokens places is the quantitative property which can be computed. In order to compute quantitative properties we will use the property of stochastic Petri nets (SPN) whereby the Markov chain of the net and the reachability graph of the underlying Place-Transition net are isomorphic. Therefore all properties of the underlying Place-Transition net also hold for SPN and vice versa. For evacuation system which is translated in term of GSPNs we will analyse the embedded Markov chain of the corresponding stochastic process [4], in other words we examine only tangible state of the system. The probability of changing from one marking to another is independent of the time spent in a marking. In this context for qualitative analysis of a GSPNs we will exploit the underlying Place - Transition net of the GSPN and use their algorithms.

There will be determined the restrictions for an integration of time, such that the results of a qualitative analysis remain valid for a quantitative analysis as well. The restrictions are related to Extended Free Choice nets, in these nets conflicts may occur between transitions of the same kind. This condition is called EQUAL-Conflict. Respecting the conditions set out above we have the following:

If we are given an $EFC-GSPN$ whose underlying Place-Transition net is live and bounded, the following holds:

- Condition EQUAL-Conflict $\Rightarrow$ GSPN has no timeless trap.
- Condition EQUAL-Conflict $\Rightarrow$ GSPN live.
- Condition EQUAL-Conflict $\Rightarrow$ GSPN has home state.

In the following we will use these relationships to determine the qualitative and quantitative properties of evacuation system in case of disaster.

4 Conclusion

In this study, a method of Generalized Stochastic Petri Nets was proposed for determining the qualitative and quantitative properties. This method allows, with some restrictions checking properties of GSPNs by means of underlying Place-Transition net.

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Section 6

Workshop

"Teaching Effectiveness Colloquium"

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Active Learning to Increase Comprehension and Retention of Applied Mathematics

James J. Cochran

Abstract

In this paper we review various definitions of active learning and distill these definitions into a single definition. We discuss some of the arguments for and against active learning, and we present an example of an active learning exercises that has been developed and used by the author with great pedagogical benefit.

Keywords: active learning, statistics, operations research, management science, education, pedagogy.

1 Active Learning Defined

The term *active learning* is very popular in the academic literature on education; it is used frequently across disciplines and it is used in a wide variety of ways. For example, Bonwell & Eison [1] define active learning as "…anything that involves students in doing things and thinking about the things they are doing." The University of Michigan Center for Research on Learning and Teaching [2] uses the definition, "…a process whereby students engage in activities, such as reading, writing, discussion, or problem solving that promote analysis, synthesis, and evaluation of class content. According to Felder & Brent [3], active learning is "…anything course-related that all students in a class session are called upon to do other than simply watching, listening and taking notes". The Center for Educational Innovation at the University of Minnesota [4] defines active learning as, "…as an approach to instruction in which students engage the material they study through reading, writing, talking, listening, and reflecting. Active learning stands in contrast to "standard" modes of instruction in which teachers do most of the talking and students are passive.”

Although this is admittedly an extremely small and nonrandom sample of definitions of active learning, these definitions do provide the reader with some understanding of the wide variety of ways that educators think about active learning. While there are indeed many stark differences in these (and other) definitions of active learning, there are also many important similarities. By focusing on these similarities, the author has developed the following definition of active learning that he uses:

> Instructional strategies and activities designed to engage students in the learning process through their participation in exercises that involve them in higher-order thinking tasks such as analysis, synthesis, and evaluation of course material.

This definition implies that the core elements of active learning are i) student engagement and participation in the learning process and ii) student involvement in higher-order thinking tasks. Furthermore, the breadth of this definition is sufficient to capture a wide range of educational activities; for example, it does not exclude exercises in which students participate outside of the classroom or work in student teams. It is also not so broad as to be meaningless.

Active learning is often contrasted with the traditional lecture approach in which students passively receive information delivered by the instructor.

2 Why Active Learning?

Many instructors ascribe to the notion that learning is a naturally active process. These instructors, generally based on their experiences as both a student and as an instructor, argue that true understanding of a complex concept is developed most effectively through student interaction with the instructor, classmates, and the concept. This position naturally leads these instructors to use of active learning strategies and exercises.

Because active learning encourages student engagement and participation in the learning process and student involvement in higher-order thinking tasks, it is thought by its advocates to

- recapture students’ attention
- emphasize critical points

- encourage higher-order thinking
- unify various concepts
- smooth the transition between major areas of coverage
- ultimately enhance comprehension and retention of complex concepts

when used effectively (see for example, Lucas et al. [5] and Freeman et al. [6]). However, not all faculty see active learning as a pedagogical panacea. Prince [7] reports that although several education researchers report evidence of the effectiveness of active learning, many educators remain skeptical and many profess to having difficulty discerning between active, collaborative, cooperative and problem-based learning.

As with any innovative approach to education, it is ultimately difficult to determine the degree of success that is due to the approach as elimination of confounding factors is impractical (Cochran, [8]). However, this author firmly believes that active learning improves student comprehension and retention of difficult concepts, but he also acknowledges that the enthusiasm he shows for his subject when executing an active learning exercise with his students may be the actual source of this improvement in student comprehension and retention of difficult concepts.

3 Teaching Matrix Transposition with an Ordinary Deck of Playing Cards – an Example of an Active Learning Exercise

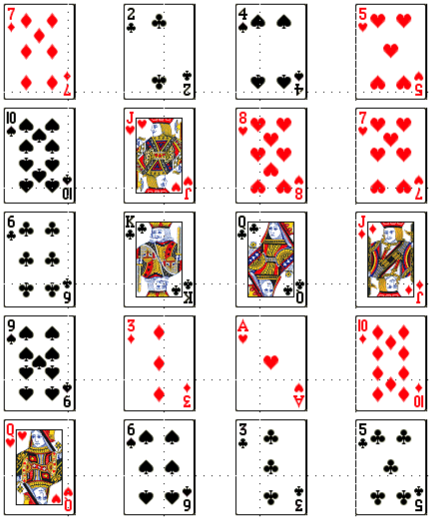
Matrix transposition, one of the first matrix operations covered in linear algebra courses, is not complicated. However, it can be difficult a difficult to comprehend concept for students who are new to matrices. Transposition is an extremely important operation that is used extensively throughout linear algebra, so it is essential that students quickly comprehend and then retain the concept.

What can an instructor to do increase the likelihood that students will quickly comprehend and then retain the concept of matrix transposition? A memorable active learning exercise may aid the instructor in the accomplishment of this pedagogical objective. The author has designed such an active learning exercise using an ordinary deck of fifty-two playing cards in a *magic trick*. The steps of the trick are:

1. Ask for a volunteer from the classroom.

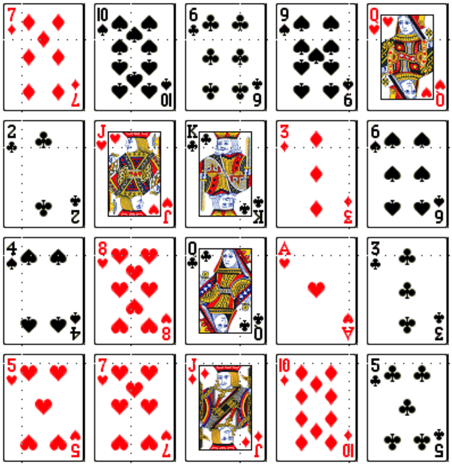
2. Select twenty cards at random from the deck and set aside the remaining thirty-two cards.
3. Deal the twenty cards into an array with five rows and four columns (for example, see Figure 1).

Figure 1 Twenty Cards in a 5x4 Array



4. Ask the student volunteer to identify the column in which the card she or he selected is located.
5. Collect the cards by column, being careful to preserve their original order of the cards.
6. Deal the cards into an array with four columns and five rows, being careful to deal the cards in a manner such that the columns in the original array are the rows in the second array, i.e., the card that was in the i^{th} row and j^{th} column in the original array is in the j^{th} row and i^{th} column in the new array (for example, see Figure 2)
7. Ask the student volunteer to identify the column in which the card she or he selected is now located.
8. Using the information the student volunteer has provided you (the column position of the selected card in the original array is the row position of the selected card in the new array), identify the card she or he selected.

Figure 2 Transposition of the Twenty Cards from the Original 5x4 Array



This simple active learning exercise engages the student and requires her or him to participate in the learning process; although the students do not realize it, during this exercise they are involved in higher-order logical and analytical thinking as they attempt to discern how the instructor was able to identify the card that was selected by the student volunteer.

This exercise takes only a few minutes of classroom time to complete, whereas it may take an instructor a substantially greater amount of time to explain matrix transposition in a more traditional manner. Furthermore, the author's experience supports the conclusion that students who have seen this demonstration comprehend and retain the concept of matrix transposition much better than those who have not. The potential return on investment for this active learning exercise is staggering.

4 Concluding Remarks

This unassuming card trick is but one example of a myriad of effective active learning exercises that have been developed by this author and others. This author has used other card tricks, television game shows, and various puzzles as the basis of active learning exercises he has developed. He has found that these exercises are effective ways to recapture students' attention, emphasize critical points, encourage students to engage in higher-order thinking, unify various concepts, smooth the transition between major areas of coverage, and enhance student comprehension and retention of complex concepts.

The instructor does have to make an initial investment of time and thought when developing an active learning exercise. This may be daunting for instructors who are new to active learning, but instructors who overcome their initial trepidation may find their investment yields tremendous payout in the long run.

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Using Hands-On Activities to Develop Students' Statistical Thinking

Roxy Peck

Abstract

This workshop will address the difference between statistical thinking and mathematical thinking. While mathematical thinking is typically developed over many years beginning in the primary grades, statistical thinking is often not introduced until a first course in statistics. This workshop will present ways to engage students in hands-on activities that are designed to develop statistical thinking and conceptual understanding.

Keywords: statistical thinking, teaching, active learning.

1 Introduction

Statistical thinking involves using data to reason and make decisions in a way that takes variability into account. Mathematical thinking can be characterized by the process of starting with a model and reasoning about what follows from that model, whereas statistical thinking involves starting with observed data and reasoning about the model that might have generated the observed data. While both mathematical and statistical thinking are important, most educational systems spend many years developing students' mathematical thinking, but devote little time to the development of statistical thinking. This means that students' first exposure to concepts that provide the foundation for statistical thinking may be in an introductory statistics course. This presents a challenge for those teaching this course because they are expected to develop students' statistical thinking in a relatively short time.

2 What is Statistical Thinking?

Although there is agreement on the importance of statistical thinking in understanding the complex world around us, there is no universally agreed upon formal definition of statistical thinking. At a minimum, statistical thinking involves reasoning with data in a way that takes variability into account. This requires an understanding of sampling variability and the role it plays in drawing conclusions from data and an understanding of the difference between convincing evidence and proof. This workshop will focus on these aspects of statistical thinking.

3 Implications for Teaching

When we ask students to think statistically, we are asking them to think in a way that is different from their previous experience with mathematics. This new way of thinking requires conceptual understanding of important concepts such as sampling variability, the difference between convincing evidence and proof, and the meaning of statistical significance. Research in statistics education has shown that direct instruction (lecture) is not very effective in developing understanding of these concepts. Activity-based instruction appears to be a more effective mode of instruction time.

This workshop will demonstrate three hands-on activities that can be used in the classroom to engage students and to demonstrate concepts that are important to statistical thinking.

4 Conclusion

Hands-on classroom activities have been shown to be an effective way to engage students, develop understanding of important concepts, and enhance students' ability to think statistically. Activities like the ones demonstrated in this workshop provide students with a level of understanding that goes beyond just procedural fluency, and that will serve as a foundation for reasoning with data to understand the world around them.

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Interactive Methods for Teaching Operations Research

Mesut Yavuz

Abstract

This talk discusses interactive teaching of Operations Research (OR). A three-phase approach to OR projects is discussed. Current students' capabilities, needs and expectations are also discussed, with particular emphasis of their implications for teaching OR. Several methods are presented to foster an interactive teaching environment, and they are also demonstrated on two examples.

Keywords: Operations research, Teaching, Spreadsheets.

1 Introduction

Operations Research (OR) is a highly interactive discipline, with roots in mathematics and computer science and vast application areas in multiple disciplines including engineering and business. In simple terms, OR can be defined as a quantitative discipline aiming to make best decisions in constrained or unconstrained settings, utilizing data and the scientific method. As such, quantifying the quality of decision alternatives (solutions) and determining what constraints may narrow the solution space make up a key part of any operations research project, called modeling.

Modeling corresponds to the problem definition step of the scientific method, which is arguably the most important step. A widely preached approach is to spend 55 minutes on defining the problem and

5 minutes on solving it. OR modeling typically results in mathematical formulation(s). Therefore, OR professionals need to have some math backgrounds. However, a math background is necessary but not sufficient. Other skills such as critical thinking and communication are also instrumental in modeling.

Once a good representative model has been built, the focus shifts to solving the model. Developing fast and accurate solution methods for OR problems is of interest to researchers in the OR, applied math, computer science and related disciplines. The practitioners are more interested in using the methods, which is typically done in two ways: (1) entering the model into a computer software, (2) implementing an algorithm in a step-by-step manner. The former is useful in developing an ability to utilize technology in tackling large-size problems. Whereas, the latter is useful in developing critical thinking skills. Algorithm implementation is more beneficial when done via spreadsheet software (e.g., Microsoft Excel). Both ways additionally develop computer literacy skills.

When a solution to the problem is obtained, the challenge of implementing the solution begins. Any shortcoming in modeling would surface during implementation, thus the modeling phase may have to be revisited. Even if the model is proven to reflect the problem accurately, the data may change at time of implementation. So, we have to ask a series of "what if" questions in an interpretation phase, following solution.

OR is the primary work area of many researchers in Industrial Engineering and Operations Management disciplines. In these disciplines, there is a tendency to express every system or method in terms of "*input* → *process* → *output*." The preceding three paragraphs roughly outline the input (modeling), process (solution) and output (interpretation) of OR projects. Some skills needed for OR (in addition to math) are critical thinking, communication, and computer literacy. In fact, these skills are sought after in the business world. Thus, OR education can help prepare students for successful careers in various fields. In this talk, some strategies for teaching OR are presented and demonstrated on two examples.

2 Understanding Today's Students' Capabilities, Needs and Expectations

One key characteristic of present day's students is that they require interaction. Teaching is historically viewed as a one-way relationship. Since the ancient times, in mentor-protégé relationships protégés have tried to capture as much knowledge from their mentors. In this structure, the mentor is the giving and the protégé is the receiving hand. While some successful applications of this model may still exist, it does not really fit the modern higher education system. Viewing teaching as the ultimate learning experience, teachers can in fact get something from teaching. Also, the more the students engage in class discussions, the more they retain, and the more enjoyable is the experience for all involved.

Another key characteristic of today's students is that they are very tech-savvy. They use their computers and smart phones, tablets, etc. very actively. However, this tech-savviness should not be confused with quantitative computation skills. A heavy computer user may be heavily under-prepared in spreadsheets coming into an OR class. Furthermore, the rigidness of formulas in spreadsheet software forms a huge barrier for students used to auto-correct.

A third important attribute is that today's students seek instant feedback. When they order some product on the Internet, they can track every movement thereof, or even get alerts on their smart phones instantly. Teachers need to keep in mind that students come to class with the same mentality. Whether it is a pop quiz, a homework or an exam, the students expect to know how they did and what the correct answer was instantly. This practically requires using some kind of a learning management system with assignments designed to be automatically graded. Luckily for us, quantitative subjects lends themselves to automatic grading.

The talk presents some strategies with applied examples to address these and some other topics.

3 Example Projects

Sports interest a large number of people. As such, the sports industry is large and provides plenty OR problems. Students relate to sports problems more easily, and, hence, they make great examples in OR classes. In this talk, a referee-assignment problem in professional football leagues is presented. In a typical European football league, the problem is quite large in size. Our focus through this example is on the modeling and interpretation phases.

Production scheduling is one of the most prolific areas of OR. Our second example considers a flow-shop scheduling problem that lends itself easily to spreadsheet modeling. Via integrated modeling and solution development in Microsoft Excel, we demonstrate several interactive teaching strategies through this example.

4 Conclusion

In this paper some proven teaching strategies are presented. Two examples, one from the field of sports and the other from production scheduling are provided as platforms for the suggested strategies.

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